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Hydrodynamic Dispersion due to a Magnetohydrodynamic-Electroosmotic Driven Flow through a Microchannel

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Abstract - In a parallel-flat plate microchannel, with nonuniform zeta potential of the wall, we analyse the dispersion of a passive solute under the simultaneous influence of electroosmotic (EOF), and magnetohydrodynamic (MHD) forces. The hydrodynamic of the flow was solved using the lubrication approximation theory (LAT) and we assume a Newtonian fluid. The solution of the electrical potential is based on the Debye-Hückel approximation for a weak potential of a symmetric ($z : z$) electrolyte solution. It is shown that the interaction between the non-uniform wall zeta potential induces a pressure gradient so as to satisfy the continuity of flow, generating a no plug like velocity profiles that contribute directly to dispersion. It is also shown that with the adding of the MHD the velocity flow increase two times its value, and the dispersion may increase more than four times as compared against the case of a purely electroosmotic forces.

1. Introduction

Transport, mixing, preparation, separation of chemical species, chemical and biological analysis are the major fluidic processes to be performed in a lab-on-a-chip. Also it's a modern practice to take blood samples and then perform a chemical analysis, using mechanism like diffusion and dispersion in the case of mixing, however the dispersion will limit the performance of chemical analysis systems such as capillary zone electrophoresis and capillary liquid chromatography with electro-osmotic. In some cases the use of magnetic fields, as non-intrusive process, is used for this analysis.

Nowadays the electrokinetic method, which mobilizes the fluid utilizing the charge density in an electric double layer, has been used widely in microfluidics as it allows us not to use mechanical mechanisms. Combined with an electric field gives rise to electro-osmotic flow, due to the contact with the interface of an electrolyte and solid surface. These devices, such as micropumps and MHD generator, are often used to pump, and control the fluid flow.[3]

It is well known that even a small amount of boundary slippage can substantially enhance electro-osmotic flow, there is also know that magnetic field will increase, or decrease, the magnitude of the velocity profile. So the aim of the present study is to determine the hydrodynamic dispersion due to the simultaneous effects of electroosmotic and magnetohydrodynamic forces of a Newtonian fluid considering that the wall potential varies in axial direction according to a sinusoidal function. Ghosal[5] solved asymptotically the hydrodynamic dispersion in electro-osmotic flow generated by the nonlinear interaction between the oscillatory wall potential and cross-section.

The Poisson-Boltzmann equation was simplified using the Debye-Hückel approximation[4].

2. Problem Definition

Fig. 1 shows the scheme of the physical model studied, corresponding a flat plate microchannel whose height is $2H$ and length L, with $H << L$. There exists a fluid flow driven by electroosmotic and magnetic forces, being E_z and B_o the strength of the externally applied electric field and the magnetic field, respectively. The upper and lower walls of the microchannel have a zeta potential, $\zeta(x)$, that varies slowly in the axial direction according to the function $\zeta(x) = \zeta_0 + \Delta\zeta Sin(\frac{2\pi x}{L})$, where $\zeta_o$is the uniform zeta potential and $\Delta\zeta$ is the zeta potential amplitude of the fluctuations.

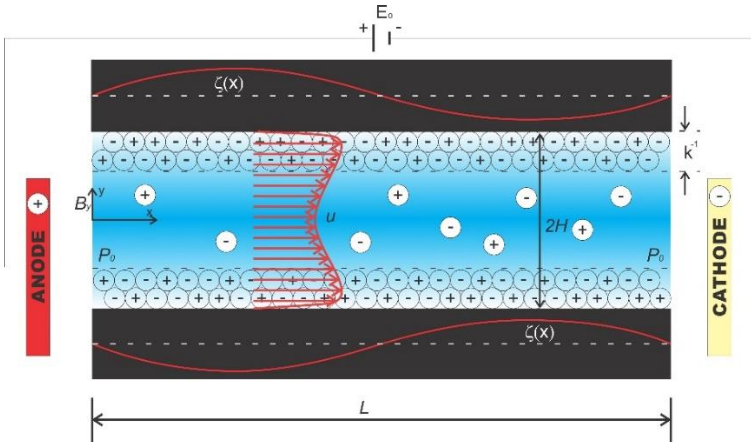

Fig. 1: Schematic diagram of the studied physical model.

As a neutral solute is considered, electrophoretic velocity is zero, and the advection velocity of the solute is the same as the fluid velocity. Both ends of the microchannels are subjected at constant pressure P_0. The fluid is considered as a laminar and incompressible steady flow.

Similarly, the component E_z is applied in the negative direction of the z-axis, generating a secondary EO force on the lateral direction of the microchannel. At this point we anticipate that this EO force can be neglected under the assumptions mentioned in our analysis. Also, the component B_0 which is applied in the positive direction of the y-axis, interacts with the electric field E_z, which produces the main MHD force in the flow direction, while the interaction between the electric field E_x produces a secondary MHD force in the lateral direction. We assumed a constant temperature T_o.

3. Governing Equations

3.1. Flow Field

The flow field is governed by the continuity and Navier-Stokes equations, given by

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

$$0 = -\nabla P + \nabla \cdot \tau + \boldsymbol{b}, \tag{2}$$

$$\tau_{xy}(y = 0) = 0 \tag{3}$$

$$\tau_{xx}(y = H) = \tau_{yy}(y = H) = 0 \tag{4}$$

$$u(y = -H, H) = v\,(y = -H, H) = 0 \tag{5}$$

respectively. Here **u** is the velocity vector **u** =u (u, v), P is the pressure, τ is the stress tensor, and **b** is the body force vector. Since we are working with microscopic scales the inertial terms aren't representative. The body force **b** that acts on the fluid is a contribution of the electrical and magnetic effects imposed on the system.

$$\boldsymbol{b} = \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}, \tag{6}$$

The first term on the right side of eq. (6) represents the Coulomb force, and the second is the Lorentz force, where **J** is the electric current density given by the Ohm's law

$$\boldsymbol{J} = \sigma_e(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}), \tag{7}$$

σ_e represents the electrical conductivity of the medium, ρ_e is the EDL charge density which is related to the electric potential field, ϕ, E is the electric field, and B is the magnetic field.

3.2. Electric Field

To define the interaction between the total electric potential and ion concentration, using the electrostatic theory, we use the Poisson equation

$$\nabla^2 \phi = -\frac{\rho_e}{\epsilon}, \tag{8}$$

where ϵ is the dielectric permittivity of the medium, and electric charge follows the Boltzmann distribution, given by

$$\rho_e = -2zen_\infty Sinh\left(\frac{ze\psi}{K_B T}\right), \tag{9}$$

where z is the valence, e is the elementary charge on an electron, n_∞ is the ionic number concentration in the neutral electrolyte, K_B is the Bolztmann constant, and T is the absolute temperature. Under the Debye-Hückel approximation, i.e., $\frac{ze\psi}{K_B T} \ll 1$, and for very large microchannels, equation is simplified as

$$\frac{\partial^2 \psi}{\partial y^2} = k^2 \psi, \tag{10}$$

$$\psi(y = H) = \zeta(x), \tag{11}$$

$$\frac{d\psi}{dy}\,(y = 0) = 0, \tag{12}$$

Where k is the Debye length, ζ is the wall potential, and ψ is the electrical potential.

3.3. Dispersion

It is known that the effective dispersion coefficient through a channel can be evaluated from[6]

$$K_{eff} = \langle \overline{Nu} - \overline{\overline{N}}\bar{u} \rangle = \langle \overline{Nu} \rangle - \langle \overline{N} \rangle \langle \bar{u} \rangle, \tag{13}$$

where N is defined by the following boundary-value problem of the Oseen type[7,8]

$$\nabla \cdot (\boldsymbol{u}N) - D\nabla^2 N = u - \bar{u}, \tag{14}$$

The dispersion in non-uniform flow is given by the Taylor-Aris theory

$$K = D\left[1 + \left(\frac{\tilde{u}_x H_*}{4D}\right)^2\right], \tag{15}$$

where K is the dispersion coefficient that describes the rate of the change in the squared variance with time, D is the molecular diffusion coefficient of the solute, $\tilde{u}_x$ is the uniform velocity field, and H_* is a length-scale parameter. As shown in equation (15) we need to determine the hydrodynamics of the problem for obtaining the dispersion coefficient.

Using the solution given by Zholkovsky[2] we can know H_* as

$$H_*^2 = \frac{64}{H} \int_0^H \left[\int_0^y F(x,y) dy \right]^2 dy, \tag{16}$$

where

$$F(x,y) = 2H \frac{u}{2 \int_{y=0}^{y=H} u dy} - 1, \tag{17}$$

and

$$Q = \frac{1}{H} \int_{y=0}^{y=H} u dy, \tag{18}$$

Q is the volumetric flow.

4. Dimensionless Mathematical Model

We non-dimensionalize the governing equations by using the following dimensionless variables:

$$\tilde{x} = \frac{x}{L} \; ; \; \tilde{y} = \frac{y}{H} \; ; \quad \tilde{u} = \frac{u}{U_{HS}} \; ; \; \tilde{v} = \frac{v\,L}{H\,U_{HS}} ; \; ; \; \Omega = \frac{E_z}{B_o U_{HS}} ; \; H_a = B_o H \sqrt{\frac{\sigma_o}{\mu_o}} \; ;$$

$$\tilde{\tau}_{xy} = \frac{\tau_{xy} H}{\mu_o U_{HS}} \; ; \; \tilde{\tau}_{xx} = \frac{\tau_{xx} L}{\mu_o U_{HS}} \; ; \; \tilde{\tau}_{yy} = \frac{\tau_{yy} L}{\mu_o U_{HS}} \; ; \; \lambda = \frac{\Delta\zeta}{\zeta_0} ;$$

$$\tilde{P} = \frac{P\,H^2}{\mu_o U_{HS} L} \; ; \Psi = \frac{ze\psi}{K_B T_o} \; ; \; \tilde{\zeta} = \frac{ze\zeta_0}{K_B T_o} \; ; \Phi = \frac{\phi}{\phi_0} \; ; \; P_e = \frac{U_{HS} H}{D}$$

$$\tilde{Q} = \frac{HQ}{2\,U_{HS} H} \; ; \; \tilde{k} = kH \; ; \; \tilde{K}_{eff} = \frac{K}{D} \; ; \; \tilde{H}_*^2 = \left(\frac{H_*}{H} \right)^2 ;$$

Where $U_{HS} = -\frac{\epsilon \zeta_0 E_x}{\mu_o}$ is the Helmholtz-Smoluchowsky velocity, μ_0 is he viscosity, P_e is the Peclet number, H_a is the Hartmann number, Ω is an electromagnetic parametric, λ is the amplitude of the $\zeta(x)$.
In microfluidic systems the parameter $\beta_1 \ll 1$ and the Reynolds number ($R_e = \frac{\rho U_{HS} H}{\mu_o}$), where R_e are typically small ($\beta_1\; O(10^{-3})$, $10^{-5} < R_e < 10^{-1}$), with this we can assume that $\beta_1 R_e \ll 1$, allowing the use of the LAT. U_{HS} is the Helmholtz-Smoluchowski velocity, ρ is the density, and μ_o the viscosity. The fluid was considered as a symmetric electrolyte solution (*z : z*), and the potential distribution, $\psi(y)$, into the electrical double layer is described by the Poisson-Boltzmann equation. The axial variation of the electric potential, $\frac{\phi_x}{L}$, is much smaller than the transverse variation of the potential into the electrical double layer. We assume a low surface potential ($\zeta_0 < 25\; mV$) within a non-overlapped EDL, the Debye-Hückel approximation is valid. Due to the symmetry of the physical model we consider only one half of it. With the magnetic field we obtain the Hartmann number which takes values from $10^{-6} < Ha^2 < 10^{-4}$, and an electromagnetic parameter Ω that takes values from $10^{-5} < \Omega < 10^{9}$. Hence, the dimensionless form of the governing equation (1), (2), (6), (7), (10), and from (15) to (18), using the lubrication approximation theory, are:

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0, \tag{19}$$

$$0 = -\frac{\partial \tilde{P}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} + \frac{1}{\tilde{\tilde{\zeta}}} \tilde{k}^2 \Psi + \Omega\, H_a^2, \tag{20}$$

$$0 = -\frac{\partial \tilde{P}}{\partial \tilde{y}}, \tag{21}$$

$$0 = \frac{\partial \Phi}{\partial \tilde{z}}, \tag{22}$$

$$\frac{\partial^2 \Psi}{\partial \tilde{y}^2} = \tilde{k}^2 \Psi, \tag{23}$$

$$\tilde{\tilde{\zeta}}(\tilde{x}) = \tilde{\zeta}\left(1 + \lambda\, Sin\,(2\pi\, \tilde{x})\right), \tag{24}$$

$$\tilde{K}_{eff} = 1 + \frac{P_e^2}{16} \tilde{Q}^2 H_*^2, \tag{25}$$

$$\tilde{H}_*^2 = 64 \int_0^1 \left[\int_o^{\tilde{y}} \tilde{F}(\tilde{x}, \tilde{y}) d\tilde{y}\right]^2 d\tilde{y}, \tag{26}$$

$$\tilde{F}(\tilde{x}, \tilde{y}) = \frac{\tilde{u}}{\tilde{\tilde{Q}}} - 1, \tag{27}$$

The dimensionless boundary conditions for the equations (19) to (23) are as follows:

$$\frac{\partial \tilde{u}}{\partial \tilde{y}}(\tilde{y} = 0) = 0, \tag{28}$$

$$\tilde{u}(\tilde{y} = -1,1) = \tilde{v}(\tilde{y} = -1,1) = 0, \tag{29}$$

$$\tilde{P}\,(\tilde{x} = 0,1) = 0, \tag{30}$$

$$\frac{d\Psi}{d\tilde{y}}\,(\tilde{y} = 0) = 0, \tag{31}$$

$$\Psi\,(\tilde{y} = 1) = \tilde{\zeta}(\tilde{x}), \tag{32}$$

The values for the physical parameters used are the representative for microchannel studies, where $H = 25\,\mu m$, $L = 0.01\,m$, $\epsilon = 10^{-10}\,CV^{-1}m$, $\sigma_o = 0.01\,Sm^{-1}$, $B_o = 1\,V\,s\,m^{-2}$, $E_z = 0.5 \times 10^3\,Vm^{-1}$, $\zeta_o = 25\,mV$, $\mu_o = 1 \times 10^{-3} kg\,m^{-1}s^{-1}$, $\rho_o = 1000\,kg\,m^{-3}$, $D_i = 3.125 \times 10^{-12}\,m^2 s^{-1}$. With these values we can calculate the dimensionless parameters: $\tilde{k} = 53644$, $\beta_1 = 0.0025$, $\lambda = 0.3$, $U_{HS} = 1.25 \times 10^{-6}$, $R_e = 3.125 \times 10^{-5}$, $H_a = 7.9 \times 10^{-5}$, $\Omega = 4 \times 10^8$, $\Omega H_a^2 = 2.5$, and $P_e = 10$.

5. Analytical Solution

Equation (23) is easily solve as a second order ordinary differential equation using the (24), (31) and (32) equations

$$\Psi(\tilde{y}) = \tilde{\zeta}\left(1 + \lambda\, Sin(2\pi\tilde{x})\right)\frac{Cosh(\tilde{k}\tilde{y})}{Cosh(\tilde{k})} \tag{33}$$

The solution of equation (20) is obtained by using equations (28) to (29), we obtain

$$\tilde{u} = \frac{1}{2}\,(\tilde{y}^2 - 1)\left(\frac{dP}{d\tilde{x}} - \Omega H_a^2\right) - (1 + \lambda\, Sin\,(2\pi\tilde{x}))(\frac{Cosh(\tilde{k}\tilde{y})}{Cosh(\tilde{k})} - 1) \tag{34}$$

using equation (19), and (34), evaluated at the boundary conditions in equations (29) and (30), we solve the pressure gradient

$$\frac{dP}{d\tilde{x}} = -3\,\lambda\, Sin\,(2\pi\tilde{x})(\frac{Tanh(\tilde{k})}{\tilde{k}} - 1) \tag{35}$$

and the mass flow rate is given as

$$\tilde{Q} = -\frac{1}{3}\left(\frac{dP}{d\tilde{x}} - \Omega H_a^2\right) - (1 + \lambda\, Sin\,(2\pi\tilde{x}))(\frac{Tanh(\tilde{k})}{\tilde{k}} - 1) \tag{36}$$

The dispersion coefficient can be obtained with incorporating equations (26) and (27) into (25), in which case we obtain

$$\begin{aligned}\tilde{K}_{eff} = 1 + \frac{P_e^2}{9\tilde{k}^2}\Bigg[& -18Sech(\tilde{k})^2(1 + \lambda\, Sin\,(2\pi\tilde{x}))^2 + \frac{9Sech(\tilde{k})^2(1 + \lambda\, Sin\,(2\pi\tilde{x}))^2 Sinh(2\tilde{k})}{\tilde{k}} \\ & + \frac{1}{105\left(1 + Cosh(2\tilde{k}) + Sinh(2\tilde{k})\right)^2}\{Cosh(2\tilde{k}) \\ & + Sinh(2\tilde{k})\}\{60\left(\tilde{k}Cosh(\tilde{k})[\Omega H_a^2 - 3\lambda Sin(2\pi\tilde{x})] + 3\lambda Sinh(\tilde{k})Sin(2\pi\tilde{x})\right)^2 \\ & + 140(\tilde{k}Cosh(\tilde{k})[\Omega H_a^2 - 3\lambda Sin(2\pi\tilde{x})] + 3Sinh(\tilde{k})[2 + 3\lambda Sin(2\pi\tilde{x})])^2 - 42(2\tilde{k}^2[\Omega H_a^2]^2 \\ & + [-27\lambda^2 + 9\tilde{k}^2\lambda^2][1 - Cos[4\pi\tilde{x}] \\ & + Cosh(2\tilde{k})[2\tilde{k}^2[\Omega H_a^2]^2 + 27\lambda^2 + 9\tilde{k}^2\lambda^2] - 36\lambda Sin(2\pi\tilde{x}) - 12\tilde{k}^2\Omega H_a^2\lambda Sin(2\pi\tilde{x}) \\ & + 12\tilde{k}Sinh(2\tilde{k})[\Omega H_a^2 - 3\lambda^2]\} - \frac{24}{\tilde{k}^4}\{1 + \lambda Sin(2\pi\tilde{x})\}\{-3\tilde{k}^2[\Omega H_a^2 - 3\lambda Sin(2\pi\tilde{x})] \\ & + \tilde{k}Tanh(\tilde{k})[3\tilde{k}^2 + 3\Omega H_a^2 + \tilde{k}^2\Omega H_a^2 - 18\lambda Sin(2\pi\tilde{x})] - 3Tanh(\tilde{k})^2[\tilde{k}^2 - 3\lambda Sin(2\pi\tilde{x})]\}\Bigg]\end{aligned} \tag{37}$$

6. Results and Discussion

In previous sections we have derived an analytical expression of the dispersion coefficient of a neutral solute which is transported in a hydrodynamic flow field in a parallel-flat plate microchannel; the fluid is driven by the simultaneous effect of electroosmotic and magnetic forces, where heterogeneous zeta potentials for the upper and bottom walls were assumed, and vary slowly and periodically in longitudinal direction. To estimate the values of the dimensionless parameters involved in the analysis, we have considered values of physical and geometrical parameters that have been reported in the specialized literature (Table 1).

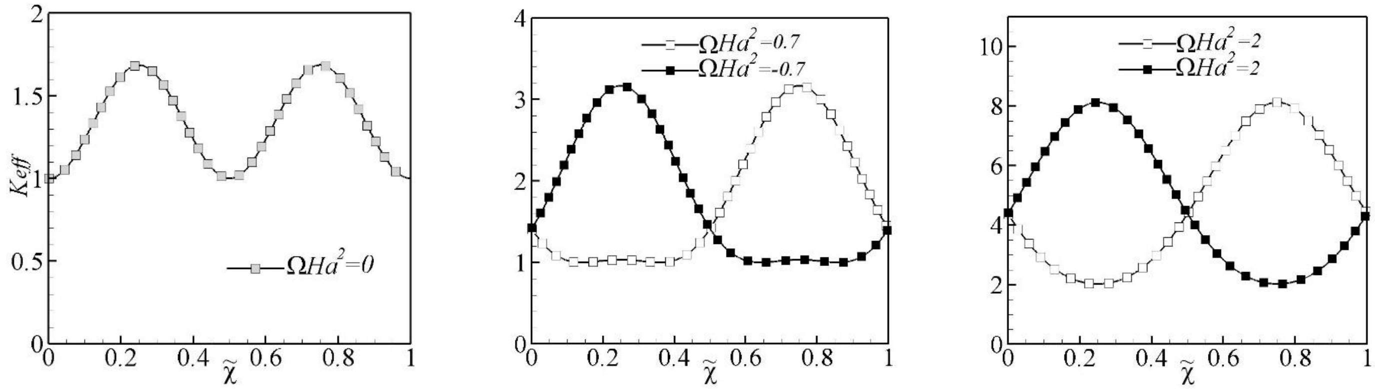

Fig. 2: Variation of the dimensionless effective dispersion coefficient $\widetilde{K}_{eff}$ with axial position caused by varying wall potential.

In Fig. 2 we have the coefficient $\widetilde{K}_{eff}$ with $P_e = 10$, $\tilde{k} = 53644$, $\lambda = 0.3$, with different values of $\Omega\, H_a^2$. We can appreciate when $\Omega\, H_a^2 = 0$ the diffusion is purely dependant of the dispersion due to the pressure gradient and the wall potential, but with the increase of $\Omega\, H_a^2$ the diffusion profile start changing due to the magnetohydrodynamic dispersion. We can also see that with a magnetic field the maximum value of $\widetilde{K}_{eff}$ exist at the entrance, when the $\Omega\, H_a^2$ is negative, and at nearly the exist, when $\Omega\, H_a^2$ is positive. In the case when $\Omega\, H_a^2$ is +/- 0.7 the increase is of 217%. When we have an $\Omega\, H_a^2$ of +/- 2 the increase is of 712%.

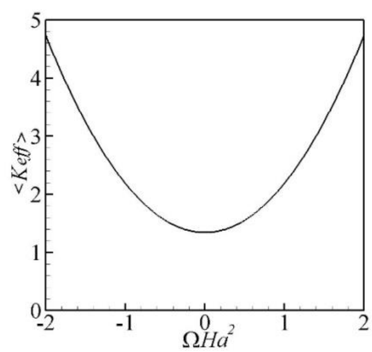

Fig. 3: Dimensionless average dispersion parameter at $\widetilde{K}_{eff}$ at $\tilde{k} = 53644$, $\lambda = 0.3$, and $P_e = 10$, with varying $\Omega\, H_a^2$.

In figure 3 we show how the average dispersion coefficient change with different values of $\Omega\, H_a^2$. With $\Omega\, H_a^2 = 0$, $\Omega\, H_a^2 = 0.7$, and $\Omega\, H_a^2 = 2$, an increase of 34%, 76%, and 373% is observe, respectively. Contrary to what was suspected, the sign of $\Omega\, H_a^2$ does not affect the average dispersion coefficient.

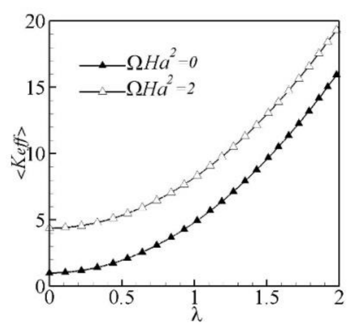

Fig. 4: Dimensionless average dispersion parameter at $\widetilde{K}_{eff}$ at $\tilde{k} = 53644$, and $P_e = 10$.

In figure 4 we observe the effects of the dimensionless average dispersion coefficient at different λ, when $\tilde{k} = 53644$, and $P_e = 10$. At $\Omega\, H_a^2 = 0$, the first remark is that $\lambda > 0.1$ so it can contribute to the increase of the average dispersion value. When $\lambda = 0.3,\ 0.5,\ 1,\ 2$ we obtain an increment of 34%, 95%, 381%, and 1524%, respectively.

When $\Omega\, H_a^2 = 2$, the increment at $\lambda = 0.3,\ 0.5,\ 1,\ 2$ are 373%, 434%, 719%, 1862%, respectively. We observe that the increments are less in percentage in comparison when we don't have magnetic fields, but it is important to consider how the magnetic fields contribute in each value of λ. In the comparison between $\Omega\, H_a^2 = 0$ and $\Omega\, H_a^2 = 2$ the increments are: at $\lambda = 0$ of 335%, at $\lambda = 0.3$ of 252%, at $\lambda = 0.5$ of 173%, at $\lambda = 1$ of 70%, and at $\lambda = 2$ of 21%.

7. Conclusion

In this work we have an analytical analysis, based on the lubrication approximation theory, of the magnetohydrodynamics and electroosmotic flow of a Newtonian fluid with a varying $\zeta(x)$ wall potential. Due to the varying wall potential we have an induced pressure gradient, which modifies the velocity profile, the stream, and the dispersion on the fluid. We also observe the effects of the inclusion of a magnetic field on the microchannel, which also modified the velocity profile (by incrementing or decreasing it), the volumetric flow rate (increasing it), the stream (increasing it and creating a sinusoidal shape), and the dispersion (increasing it).

We suggest the use of a rheological model for further analysis of the dispersion process with a variable wall potential, as well as the implications of MHD and EO flows variable in time.

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References

[1] A. Ajdari, "Generation of transverse fluid currents and forces by an electric field: Electro-osmosis on charge-modulated and undulated surfaces," *Phys. Rev*., vol. 53, 1995.

[2] E. K. Zholkovskij, J. H. Masliyah, A. E. Yaroshchuk, "Broadening of neutral analyte band in electroosmotic flow through slit channel with different zeta potentials of the walls," *Microfluid Nanofluid*, vol. 15, pp. 35-47, 2013.

[3] J. Escandon, F. Santiago, O. Bautista, F. Mendez, "Hydrodynamics and thermal analysis of a mixed electromagnetohydrodynamic-pressure driven flow for Phan-Thien-Tanner fluids in a microchannel," *International Journal of Thermal Sciences*, vol. 86, pp. 246-257, 2014.

[4] J. Masliyah and S. Bhattacharjee, "Electrokinetic and Colloid Transport Phenomena," Wiley-Interscience, 2006.

[5] S. Ghosal, "Lubrication theory for electro-osmotic flow in a microfluidic channel of slowly varying cross-section and wall charge," *J. Fluid Mech*., vol. 459, pp. 103-128, 2002.

[6] C.-O. Ng, Q. Zhou, "Dispersion due to electroosmotic flow in a circular microchannel with slowly varying wall potential and hydrodynamic slippage," *Physics of Fluids*, vol. 24, 2012.

[7] C.-O. Ng, B. Chen, "Dispersion in Electro-Osmotic Flow Through a Slit Channel With Axial Step Changes of Zeta Potential," *J. Fluids Eng.-Trans. ASME*, vol. 135, pp. 101203-1--101203-8, 2013.

[8] Qi Zhou, "Electro-osmotic dispersion in a circular tube with slip-stick striped wall," *Fluid Dyn. Res*., vol. 47, pp. 015502, 2015.