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A Modified Preconditioning Approach for Nodal Integral Method

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Abstract **-** Nodal Integral Methods (NIM) are numerical methods which solves partial differential equations in quite efficient and accurate manner. The available conventional methods such as Finite Difference Method (FDM) or Finite Volume Methods (FVM) need very fine mesh compared to NIM to attain the same level of accuracy. The high accuracy of NIM is due to the use of semi analytic solutions of approximate ODEs for each node in the mesh. In spite of significant merits over other methods, the prevailing use of NIM is limited only for linear or weakly nonlinear problems. Recently, some efforts have been made to extend the application of NIM for higher non-linearity by using Jacobian-Free Newton Krylov (JFNK) approach which is an efficient implementation of the Newton method. A Preconditioning algorithm of JFNK is given for Burger's equation in both dimension in 1D and 2D, but the work was not extended for higher nonlinearity because of the singularities in the coefficients at higher Reynolds numbers. In the present study, some modifications in the coefficients are given to extend the algorithm for relatively high nonlinearity compared to earlier work. The numerical results are compared with the analytical solution to demonstrate the accuracy of the proposed algorithm. Furthermore, the spectral analysis is performed to test the eigenvalues clustering ability of the proposed algorithm.

Keywords**:** Nodal integral method, Jacobian free Newton Krylov method, Burgers' equation

1. Introduction

Real-life problems describing the physical phenomena are generally represented by nonlinear Partial Differential equations (PDE) and the solution of these PDEs has always been a difficult task for the researchers in the past. Although many researchers in the literature have given a list of methods such as Finite Difference Method (FDM) or Finite Volume Methods (FVM) and many more to solve these PDEs but still the efficient and accurate solution of these equations is a big challenge so far. The three above-mentioned point-based methods are very popular and are widely used technique to solve nonlinear PDEs because they provide reasonably good accuracy of the solution. The main concern with those methods is that they need very fine mesh to attain the good accuracy therefore need more efficient schemes. To overcome the requirement of fine mesh, coarse mesh methods were developed in the late 1960's. The main motivation behind the development of these coarse mesh methods were to solve the PDEs by using coarser grids instead of using the fine grids as other conventional methods do. Among the various types of coarse mesh methods, Nodal methods were developed to solve the problem related to nuclear industry such as neutron diffusion equation. The concept of nodal method emerged from the Partial current balance method [1] and nodal green's function method [2]. Dorning [3] and Lawrence [4] then presented a detailed review of the nodal method. Also, the relevance between the nodal methods and point-based methods such as FDM was given by Hennart [5]. The success of nodal methods in reactor kinetics has motivated its extension to fluid flow problems and for the very first time, Horak [6] and Doring [7] developed nodal methods for the solution of incompressible Navier Stokes equation by using nodal green's function approach. Later, Azmy [8] found that similar results can be obtained by the Nodal Integral Method (NIM), which is conceptually simpler than the nodal method and requires fewer discrete unknowns per node.

In all nodal-based approaches, the transverse integration process (TIP) is the primary step. TIP converts the PDE into a set of approximate ODEs. These ODEs are then solved analytically which a node which leads to linear, quadratic, exponential and trigonometric coefficients-based solutions according to the reduced governing equations and hence capture the actual physical aspects of the system in greater detail. Due to this semi-analytic approach of NIM, it gives very accurate and efficient

solution by using coarser grid instead of using the fine grids as other conventional methods do. Despite such merits over other methods the prevailing use of NIM is limited only for linear or weakly non-linear problems because so far people have used Picard-type solvers for the solution of the NIM scheme. Since Picard-based solvers can perform only linear iterations, these solvers cannot unlock the full potential of the scheme; therefore, better solvers are needed to carry out the simulation with NIM. Recently, an approach called JFNK_NIM with physics-based preconditioner has been developed for Burgers equation for both one-dimensional (1D) and two dimensional (2D) problems [9,10]. In case of 2D, the reported work is limited up to Reynolds number of 2500, which though exhibits high non-linearity but due to singularities in coefficients, NIM cannot solve beyond aforementioned the Reynolds number. Therefore, these singularities need to be removed for a possible extension of NIM for high Reynolds number. In the present work an approach has been proposed to remove singularities from these coefficients. In order to demonstrate the robustness of the scheme the solutions are obtained till very high Reynolds number (i.e., up to 35000). To test the accuracy of NIM at high Reynolds number, the NIM results are compared with the exact solution. Furthermore, the spectral analysis of the preconditioner matrix has also been carried out to understand the effectiveness of the proposed algorithm. The analysis reveals that the proposed preconditioning is quite effective in clustering eigenvalues at high Reynolds number. Therefore, it can be concluded that the algorithm is very robust and effective in handling high nonlinearity, which can be further extended to the solution of Navier–Stokes equation in a straightforward manner.

2. Nodal Integral Method

Nodal Integral Method can be classified in various steps. A brief overview of development of NIM is given in four essential steps as follows.

First the discretization of space-time domain is divided into finite size brick like elements also called here as nodes. Each node is of width $\Delta x=2a$, length $\Delta y=2b$ and height $\Delta t=2\tau$ as shown in figure. After discretization, Transverse Integration Process (TIP) is applied over each node which converts each PDE in a set of ODEs. The averaged variables in these ODEs are called as transverse-integrated variables. The set of ODEs are then split in to homogenous as well as inhomogeneous parts. The part of equation which can be easily integrated is taken as homogenous on the left-hand side of the differential equation whereas the part which cannot be easily integrated is referred to as inhomogeneous part and taken on right hand side of differential equation. The inhomogeneous part popularly known as pseudo source term and solutions are obtained for this term in the form of Legendre polynomial expansions that are truncated at a particular order (here at zero order). The homogenous term within each cell is integrated and solved for transverse averaged variables in terms of the pseudo source terms. The analytical solution of the ODE's lead to linear, quadratic, exponential, trigonometric solutions according to the reduced governing equations and hence capture the actual physical aspects of the system in greater detail. After that the condition of continuity of each transverse averaged variable (and their derivatives for the second order of ODEs) are then applied at the node interface over each cell in order to obtain the coupled equation between two adjacent nodes called here as three-point scheme. The pseudo source terms appear as unknowns after truncation and additional conditions are required to completely attain the explicit solutions. Since the pseudo source terms are the extra unknowns and were never evaluated formally therefore need to be eliminated. Hence some physically relevant constraint conditions are then used to eliminate pseudo source terms and obtain a number of algebraic equation equal to the number of discrete unknown per node.

3. Formalism

Burgers equation has been used by several researchers as a benchmark problem for accurate comparison of their numerical scheme. Two-dimensional Burgers' equation is given as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{Re}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] \quad (1)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{1}{Re}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right] \quad (2)$$

Here u and v are the velocity in x and y directions, respectively and 'Re' represent the Reynolds number.

Several modifications in NIM have been performed to make it more agile, efficient and accurate. Modified Nodal Integral Method (MNIM) and Modified MNIM (M^2NIM) are the two commonly used method in the fluid flow

community. MNIM is more implicit and gives more accurate solution whereas (M^{2}NIM) have some delayed effect in it therefore gives faster convergence. In this earlier work [10], the Jacobian free Newton-Krylov (JFNK) approach was used to used to solve Burgers' equation using MNIM. Moreover, Krylov solvers do not generally work well without an appropriate appropriate preconditioner. Therefore (M^{2}NIM) was used as a preconditioner to accelerate the solution of MNIM which maintains both, speed as well as the accuracy. In the present study the main focus is to extend the earlier work for high nonlinearity not to repeat the same again. The complete discussion and the development process of the algorithm can be found in Niteen and Singh [10] work. Here, only the final set of algebraic equation are shown because the concern of this work is to remove the singularities from the coefficients in the final equation. Therefore, after applying all the steps discussed in the previous section the final set of algebraic equations are given as,

$$F_{37}\bar{u}_{i,j}^{xt} = F_{31}\bar{u}_{i,j-1}^{xt} + F_{32}\bar{u}_{i,j+1}^{xt} + F_{33}\left(\bar{u}_{i,j,k}^{xy} + \bar{u}_{i,j,k-1}^{xy}\right) + F_{34}\left(\bar{u}_{i,j+1,k}^{xy} + \bar{u}_{i,j+1,k-1}^{xy}\right) \tag{3}$$

$$F_{57}\bar{u}_{i,j}^{yt} = F_{51}\bar{u}_{i-1,j}^{yt} + F_{52}\bar{u}_{i+1,j}^{yt} + F_{53}\left(\bar{u}_{i,j,k}^{xy} + \bar{u}_{i,j,k-1}^{xy}\right) + F_{54}\left(\bar{u}_{i+1,j,k}^{xy} + \bar{u}_{i+1,j,k-1}^{xy}\right) \tag{4}$$

$$F_{37}\bar{v}_{i,j}^{xt} = F_{31}\bar{v}_{i,j-1}^{xt} + F_{32}\bar{v}_{i,j+1}^{xt} + F_{33}\left(\bar{v}_{i,j,k}^{xy} + \bar{v}_{i,j,k-1}^{xy}\right) + F_{34}\left(\bar{v}_{i,j+1,k}^{xy} + \bar{v}_{i,j+1,k-1}^{xy}\right) \tag{5}$$

$$F_{57}\bar{v}_{i,j}^{yt} = F_{51}\bar{v}_{i-1,j}^{yt} + F_{52}\bar{v}_{i+1,j}^{yt} + F_{53}\left(\bar{v}_{i,j,k}^{xy} + \bar{v}_{i,j,k-1}^{xy}\right) + F_{54}\left(\bar{v}_{i+1,j,k}^{xy} + \bar{v}_{i+1,j,k-1}^{xy}\right) \tag{6}$$

$$F_{77}\bar{u}_{i,j}^{xy} = F_{71}\bar{u}_{i,j}^{xt} + F_{72}\bar{u}_{i,j-1}^{xt} + F_{73}\bar{u}_{i,j,k-1}^{xy} + F_{74}\bar{u}_{i,j}^{yt} + F_{75}\bar{u}_{i-1,j}^{yt} \tag{7}$$

$$F_{77}\bar{v}_{i,j}^{xy} = F_{71}\bar{v}_{i,j}^{xt} + F_{72}\bar{v}_{i,j-1}^{xt} + F_{73}\bar{v}_{i,j,k-1}^{xy} + F_{74}\bar{v}_{i,j}^{yt} + F_{75}\bar{v}_{i-1,j}^{yt} \tag{8}$$

where all F's are the coefficients that contains both linear as well as exponential terms in it with respect to the present time velocities u_0 and v_0. For example,

$$A_{91} \equiv \frac{-\frac{1}{Re}+\frac{b_j v_{0i,j}\left(1+e^{Rev_{i,j}}\right)}{-1+e^{Rev_{i,j}}}}{{v_{0i,j}}^2},$$

$$A_{92} \equiv \frac{e^{Rev_{i,j}}}{-1+e^{Rev_{i,j}}} - \frac{1}{Rev_{i,j}},$$

$$F_{72} \equiv -\frac{A_{92}}{A_{91}},$$

4. Coefficients Singularities

The earlier work was limited up to Re = 2500 and was not extended for Reynolds number higher than that due to the singularities in the coefficients. All the coefficients consist of exponential as well as some linear terms with respect to the local Reynolds number ($Rev_{i,j}$). These local Reynolds number depends on node averaged velocities (u_0 and v_0) therefore, it is very important to take care of node averaged velocities very carefully to make the NIM more flexible and more robust.

To understand the above-mentioned problem, a coefficient A_{92} is taken which is given as

$$A_{92} \equiv \frac{e^{Rev_{i,j}}}{-1+e^{Rev_{i,j}}} - \frac{1}{Rev_{i,j}} \tag{9}$$

where $Rev_{i,j} = dx * Re * v_0$ is the local Reynolds number.

If v_0 is zero then the last term on the right-hand side of above Equation (i.e. $\frac{1}{Rev_{i,j}}$) will become infinite by which the coefficients will blow up and solution will diverge. Therefore, applying limiting condition is necessary at this point which was already applied in earlier work but taking only limit will not be sufficient for the cases of higher Reynolds number. The series expansion of the complete coefficient up to 3rd or 4th order is necessary to maintain the robustness of the scheme. The 4th order approximation of the above coefficient is given as

$$A_{92} \equiv \frac{1}{2} + \frac{Rev_{i,j}}{12} - \frac{{Rev_{i,j}}^3}{720} \tag{10}$$

Similarly, one can easily get the series expansion of other coefficients in the same manner by using Wolfram Mathematica.

On the other hand, two more important thing should also be noted, first the zero condition is checked on the basis of local Reynolds number ($Rev_{i,j}$) not on the node averaged velocities (v_0) and secondly the zero condition should be in between the range ($-0.0001 < Rev_{i,j} < +0.0001$).

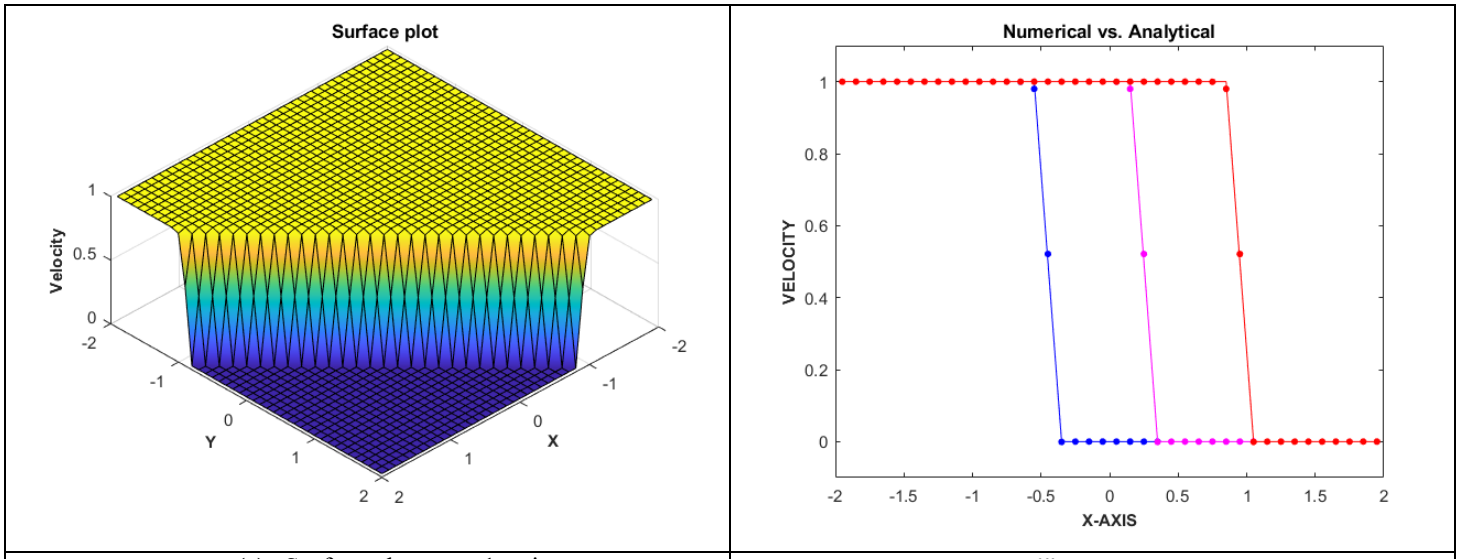
The exponential terms will also create the problems in some cases whenever $Rev_{i,j}$ becomes infinitely large. To handle this issue one can easily expand the series at infinity in a similar manner as done in case of zero velocity.

5. Results and Discussion

In this section, the solutions are obtained for 2D Burgers' equation for different Reynolds number at different grid sizes. The comparisons of krylov iterations and clustering of eigenvalues are shown to demonstrate the effectiveness of the present algorithm.

5.1. Verification of Results

It is to be noted that the detailed analysis of MNIM and PC-MNIM has been previously reported in the work of Niteen and Singh [10] but that work was limited up to Reynolds number 2500.

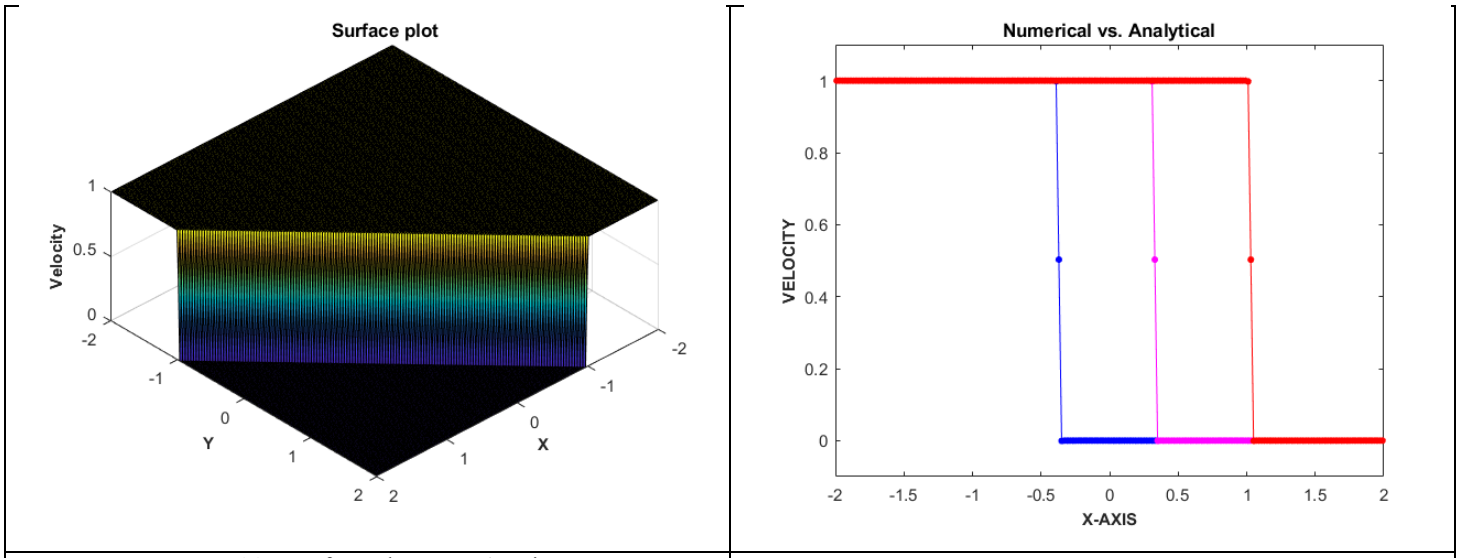


(a) Surface plot at t = 1-unit

(b) Velocity $(\bar{u}^{xy})$ at three different time steps

Fig. 1: Numerical solution obtained using PC-MNIM For Reynolds Number 5000

In the present work, the results for higher Reynolds number (Re = 5000 and 35000) are shown to demonstrate the effectiveness, as well as the stability of the present algorithm. The results are given for two different Reynolds number (Re = 5000 and 35000) at two different grid size (dx=40 and dx=200) respectively, in terms of surface plot and 2D line plot. The surface plot is given just to show the smooth wave pattern at t=1-unit time. The 2D line plot is given to compare the numerical versus analytical solution of the velocity $(\bar{u}^{xy})$ at three different time steps (t=0.3-unit (blue), t=1-unit (magenta) and t=1.7-unit (red)).



(a) Surface plot at t = 1-unit

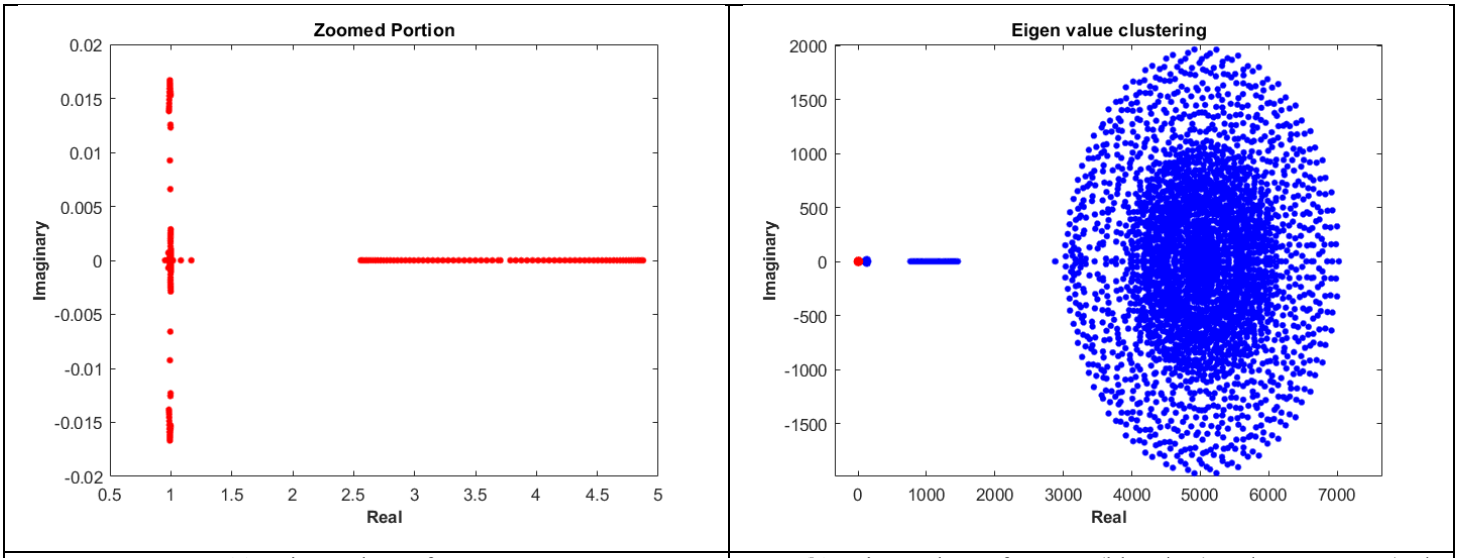
(b) Velocity $(\bar{u}^{xy})$ at three different time steps

Fig. 2: Numerical solution obtained using PC-MNIM For Reynolds Number 35000

It is to be noted that all the dots represent the numerical solutions and solid line represents analytical solutions. FIGURE 1 shows the results at Re = 5000, dx=0.1 and dt=0.01 whereas FIGURE 2 shows for Re = 35000, dx=0.01 and dt=0.01. In both the cases results matches very well with the analytical solution which demonstrate that the present algorithm is stable and accurate in providing the solution for very high Reynolds number.

5.2. Spectral Analysis of the System

The preconditioning of JFNK method is said to effective if the eigenvalues of product $\boldsymbol{JP^{-1}}$ (called as preconditioned matrix system) cluster around 1. The clustering capacity of the present algorithm is shown in FIGURE 3. It is observed that, with increasing the Reynolds number the eigenvalues spread more and more for the non-preconditioned system which results in a slow convergence. On the other hand, in case of preconditioned system faster convergence is achieved due to clustering of eigenvalues around 1. To show the comparison between preconditioned (PC-MNIM) and non-preconditioned system (MNIM), taken a case of extremely high Reynolds number 35000 and plotting the eigen values for both cases in one plot as shown in FIGURE 3 (a). It is to be noted that all the blue dots represent the eigenvalues of MNIM whereas all red dots (highlighted in a small circle in the left side of FIGURE 3 (a)) represent the eigenvalues of PC-MNIM. Since the eigenvalues for the preconditioned matrix system are very small therefore, the real pattern of eigenvalues of MNIM can only be seen in the zoomed portion as shown in FIGURE 3 (b).



(a) Eigenvalues of PC-MNIM

(b) Eigenvalues of MNIM (blue dots) and PC-MNIM (red dots)

Fig. 3: Eigenvalues clustering for Reynolds number 35000

The comparison of krylov iteration for MNIM as well as PC-MNIM is given in TABLE 1. It is shown that upon using the preconditioning algorithm the number of Krylov iterations reduces drastically as increasing the Reynolds number. Different Reynolds number have different grid size and these grid sizes are chosen in such a way that no less than this size could be possible. The time step size is taken constant for all the cases just to show the clear vision in the reduction. The tolerance of Newton as well GMRES is taken 10^{-6} and all the calculation are done on MATLAB 2020b and simulation is carried out on Intel(R) Core (TM) i7-7700 CPU.

Table 1: Comparisons of krylov iterations for MNIM and PC-MNIM at (dt = 2 τ = 0.01).

Reynolds number (Re)	Grid size (n=m)	Krylov iterations per time step		Iterations Reduction Factor
		MNIM	PC-MNIM	
5000	40	1258	36	34.94
10000	60	3058	54	56.63
15000	90	4779	75	63.72
20000	120	8378	107	78.29
25000	150	11380	128	88.90
30000	180	16329	154	106.03
35000	200	23775	182	130.63

6. Conclusion

The singularities in the coefficient are removed to extend the work established earlier. Results for higher Reynold's number are then compared with the analytical solution and it is found that the algorithm is stable at higher Reynolds number even with large time step and quite coarse grid. Furthermore, a comparison of the krylov iterations for both MNIM as well as PC-MNIM is given to demonstrate the effectiveness of the proposed algorithm. On the other hand, spectral analysis is also given for PC-MNIM versus MNIM at Re = 35000 which shows the ability of the algorithm to cluster the eigenvalues in a very effective manner as can be seen in the relevant results.

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