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Mathematical Modelling of Forced Convection: Insights from Hele-Shaw Analogues and Cylinder Studies

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Abstract – Despite being discovered more than a century ago, there is still a lack of understanding regarding the porous media analogue of the Hele-Shaw cell. It offers visualization of fluid flow and enhances knowledge of heat transfer in various technical applications. This study introduced a mathematical model of forced convection around a cylinder within a Hele-Shaw cell. The equations governing forced convection of porous media are modelled using the continuity equation, Navier-Stokes equation, and conservation of energy equation, under reasonable assumptions. Non-dimensionalization was carried out using the Buckingham Pi Theorem. The mathematical model was solved using the CFD software ANSYS Fluent, and the results were validated against available experimental and theoretical data from the literature. Simulation of forced convection over a cylinder in a Hele-Shaw cell we performed. The results showed that the Reynolds number has the direct impact on the Nusselt number. In general, as the Reynolds number increase, the Nusselt number also increases. This shows that the convection around the cylinder becomes more efficient when higher Reynolds numbers are applied to the system.

***Keywords*:** Hele-Shaw Cell, Mathematical Modelling, Forced Convection, Porous Media

1. Introduction

Research on fluid flow and heat transfer in porous media has been conducted for many years because of its practical applicability. Applications include geothermal reservoirs, underground water flow, solar collectors [1], packed-bed reactors, catalytic and chemical particle beds, solid-matrix heat exchangers [2], tumour growth modelling [3], cooling of electronic devices, as well as various medical and biological issues. One of the main concerns about the porous medium is the fluid behaviours and convective heat transmission inside the system.

The Hele-Shaw cell is a device consisting of two closely spaced parallel plates, designed for modelling porous media [4]. The Hele-Shaw model was initially created to analyze the potential flow around bodies of different shapes [5]. Although the Hele-Shaw cell was discovered more than a century ago, studies on porous media analogies are scarce, both in the theoretical and numerical aspects. The convection around a cylinder embedded in a Hele-Shaw cell is an intriguing and significant topic with high practical potential, but research in this area has been found to be lacking. Many studies have been conducted on convection from a cylinder within a porous medium, but few precisely in a Hele-Shaw cell. This study presents the mathematical modelling of forced convection around a heated cylinder enclosed in a Hele-Shaw cell, which is anticipated to provide insights into the topic and the fundamentals needed for future studies involving Hele-Shaw cell.

2. Problem Statement

A cylinder is positioned horizontally in a vertical Hele-Shaw cell and then heated. Assumptions are made at the start of the modelling process. It is assumed that the fluid is incompressible, the flow is in laminar form, the temperature of the fluid is below boiling point and the properties of the fluid, including thermal conductivity, viscosity, specific heat and thermal expansion coefficient, are constant.

The temperature (T) and pressure (P) are considered to change across the small thickness of the Hele-Shaw cell (h).

$$T = T(x, y), \qquad P = P(x, y)$$

The 3-dimensional conventional continuity, Navier-Stokes (momentum), and energy equations derived from the previously mentioned assumptions are provided as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\frac{-\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right), \tag{2}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=\frac{-\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right), \tag{3}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right) \tag{4}$$

where u, v and w are the velocity profiles in the x, y and z direction, p is the pressure, T is the temperature of the fluid, ρ is the density, c_p is specific heat, k is thermal conductivity, and υ is the kinematic viscosity.

As per Zhak et al [6], h is significantly lower than the radius of the cylinder (r), i.e., $h \ll r$, as depicted in Fig. 1. Due to the confined space between the wall and the z-axis, it is assumed that there is no motion along the z-axis in the Hele-Shaw cell. This leads to the realization of the Poiseuille profile for both longitudinal and transverse velocity profiles in a porous medium, as depicted below, with the velocity profile w=0.

$$u(x,y,z)=\frac{3}{2}u_0\left(1-\frac{4z^2}{h^2}\right), \tag{5}$$

$$v(x,y,z)=\frac{3}{2}v_0\left(1-\frac{4z^2}{h^2}\right). \tag{6}$$

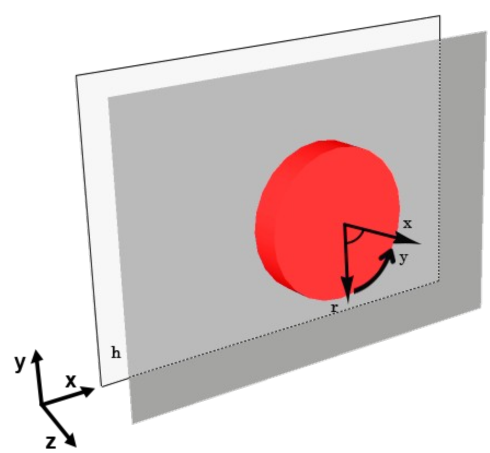

Fig. 1: Schematic figure of a heated cylinder embedded in Hele-Shaw cell

Next, substitute equations (5) and (6) into equations (1), (2), (3), and (4). The system will be integrated throughout the z-direction from $z=\frac{+h}{2}$ to $z=\frac{-h}{2}$. The three-dimensional set of equations will be simplified to a two-dimensional momentum and heat transfer phenomenon in a Hele-Shaw cell at the centre plane z=0.

$$\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}=0, \tag{7}$$

$$\frac{6}{5}\rho\left(u_0\frac{\partial u_0}{\partial x}+v_0\frac{\partial u_0}{\partial y}\right)=\frac{-\partial p}{\partial x}+\mu\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2}\right)+\frac{12\mu u_0}{h^2}, \tag{8}$$

$$\frac{6}{5}\rho\left(u_0\frac{\partial v_0}{\partial x}+v_0\frac{\partial v_0}{\partial y}\right)=\frac{-\partial p}{\partial y}+\mu\left(\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 v_0}{\partial y^2}\right)+\frac{12\mu v_0}{h^2}, \tag{9}$$

$$u_0\frac{\partial T}{\partial x}+v_0\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right), \tag{10}$$

The last terms in equations (8) and (9) are diffusion terms obtained from z-axis diffusion terms. Equations (7), (8), (9), and (10) represent the system of equations governing forced convection in the Hele-Shaw cell.

Dimensional analysis was done on the governing equation, which led to the following function:

$$Nu=f\left(\Re, Pr, \frac{h}{D}\right) \tag{11}$$

Where *D* is the diameter of the cylinder, *Re* is the Reynolds number ($\frac{\rho VD}{\mu}$) and *Pr* is the Prandtl number ($\frac{c_p\mu}{k}$).

As the dimensionless variables are defined as following:

$$X=\frac{x}{D}, Y=\frac{y}{D}, U=\frac{u_0}{u}, V=\frac{v_0}{v}, H=\frac{h}{D}, \theta=\frac{T-T_0}{T_w-T_0},$$

where u and v are the upstream velocity in *x* and *y* directions, T_0 is the bulk temperature, and T_w is the wall temperature.

H is the ratio of Hele-Shaw cell thickness (*h*) to the diameter of the cylinder (*D*), as shown in equation (11) using dimensionless variables. Therefore, equation (11) may be rewritten as

$$Nu=f(\Re, Pr, H) \tag{12}$$

The non-dimensional governing equations as given below:

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0, \tag{13}$$

$$\frac{6}{5}\left(U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right)=\frac{-\partial P}{\partial X}+\frac{1}{\Re}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)+\frac{1}{\Re}\frac{12U}{H^2}, \tag{14}$$

$$\frac{6}{5}\left(U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}\right)=\frac{-\partial P}{\partial Y}+\frac{1}{\Re}\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)+\frac{1}{\Re}\frac{12V}{H^2}, \tag{15}$$

$$U\frac{\partial \theta}{\partial X}+V\frac{\partial \theta}{\partial Y}=\frac{1}{RePr}\left(\frac{\partial^2 \theta}{\partial X^2}+\frac{\partial^2 \theta}{\partial Y^2}\right). \tag{16}$$

The equations provided involve four unknowns: velocity in the X-axis (U), velocity in the Y-axis (V), pressure (P), and temperature (θ). The Darcy number (Da) is defined as

$$Da=\frac{K}{d^2} \tag{17}$$

where K is the permeability of the medium, and d is the characteristic length, in this case, the diameter of the cylinder, D. The permeability of the Hele-Shaw cell is $\frac{h^2}{12}$ [5][8]. Substitute into equation (17), thus

$$Da=\frac{h^2}{12D^2}. \tag{18}$$

Hence, the dimensionless Darcy number can become

$$Da=\frac{H^2}{12}. \tag{19}$$

Substitute the Darcy number into equations (14) and (15), it becomes

$$\frac{6}{5}\left(U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right)=\frac{-\partial P}{\partial X}+\frac{1}{\Re}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)+\frac{U}{ReDa}, \tag{20}$$

$$\frac{6}{5}\left(U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}\right)=\frac{-\partial P}{\partial Y}+\frac{1}{\Re}\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)+\frac{V}{ReDa}. \tag{21}$$

Hence, the set of governing equations to be solved are given by equations (13), (16), (20) and (21).

3. Numerical Setup

The governing equations will then be solved using CFD software ANSYS Fluent. The boundary conditions for non-dimensional governing equations above as shown in Table 1:

Table 1: Non-Dimensional Boundary Conditions

	U	V	T
Inlet	$U_{\infty}=1$	0	$T_{\infty}=0$
Walls	0	0	0
Outlet	0	0	0
Cylinder	0	0	$T=1$

The simulation domain is modelled after the experimental geometry of Zhak et al [6], which was used to investigate mass transfer for a similar physical event. DesignModeler, an embedded tool within ANSYS, was utilized for creating the geometry. A rectangular plane was sketched, and a small cylinder with a diameter of D was positioned on the plane. Fig. 2 shows the geometry used in the current simulation.

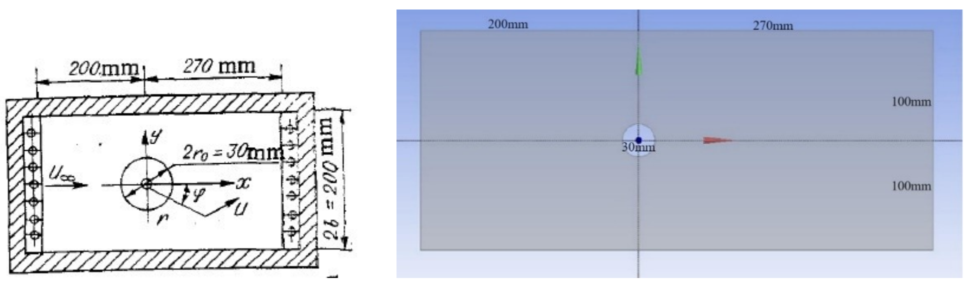

Fig. 2: The experimental setup by Zhak et al. [6] (Left), the plane geometry pf a 2-D Hele-Shaw cell with a cylinder created using DesignModeler (Right)

Fig. 3 shows the close-up mesh of the computing domain around the cylinder. The All Triangles Method was selected for the mesh element shape. In the meshing setup, Inflation was added to the cylinder that was placed as the boundary edge. The maximum number of layers was set to 2, and the growth rate was set at 1.2. Edge sizing was applied at the cylinder boundary edge, specifying the sizing type as the number of divisions.

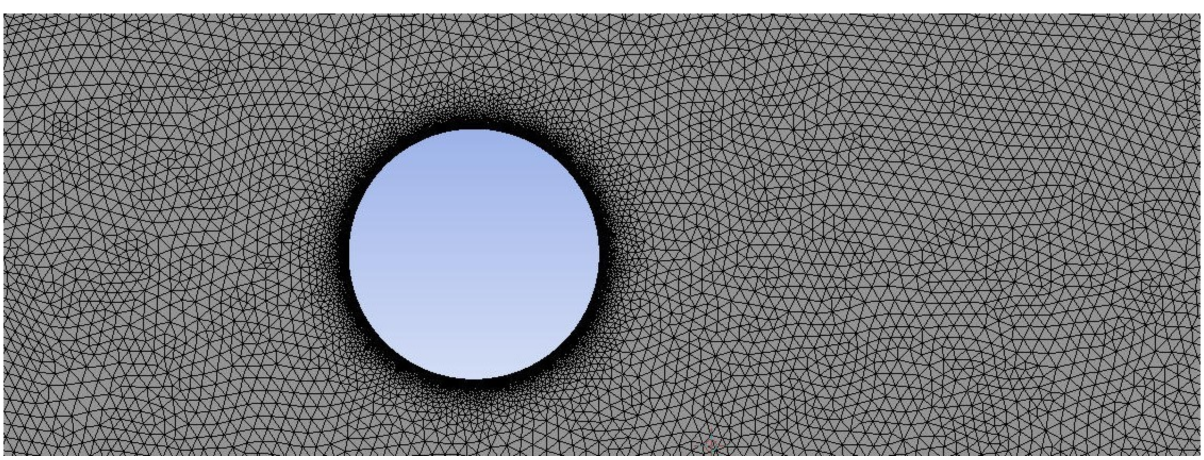

Fig. 3: Close-up mesh around the cylinder in the computation domain

4. Validation

The validation was conducted by comparing the coefficient of friction (C_f) at the stagnation point of the cylinder in reference [7] with the findings obtained from the simulations. In Fig. 4, it is shown that there is a good agreement between these results, where trendlines behave similarly, especially between calculation and current studies.

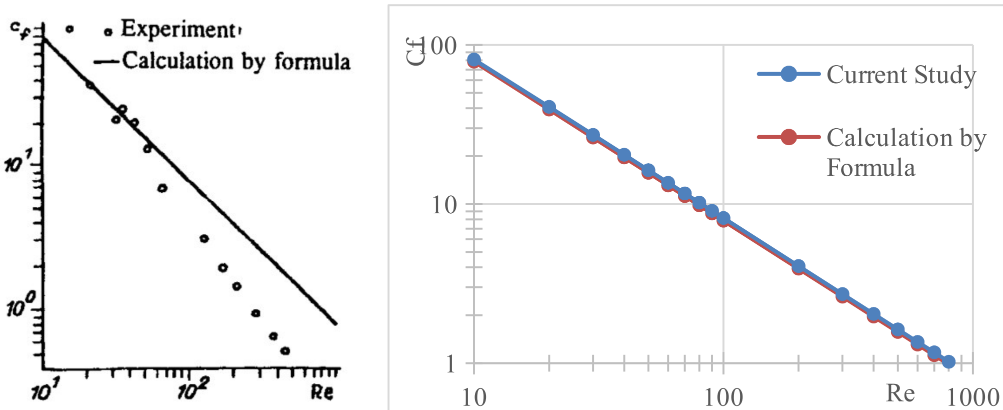

Fig. 4: Comparison of the results found in [7] (Left), with the results obtained in the current study including calculation by formula and simulations results (Right)

5. Results

The system of equations representing the forced convection around a cylinder placed inside a Hele-Shaw cell was solved for Reynolds numbers ranging from 100 to 5000. The temperature difference contour images for the forced convection around a heated cylinder contained in a Hele-Shaw cell at Reynolds numbers of 100 and 5000, respectively, are shown below in Fig. 5. A temperature plume line that extends from the cylinder's rear to its far back can be seen in all the figures. Even though the figure produced comparable findings, it is noticeable that the plume line thickens with increasing Reynolds number.

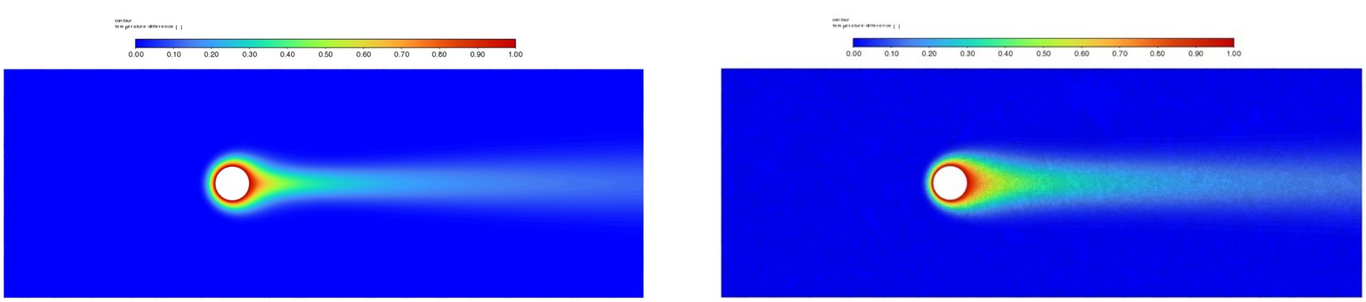

Fig. 5: The temperature difference of forced convection around a heated single cylinder embedded in Hele-Shaw cell at Reynolds number 100 (left) and 5000 (right)

Fig. 6 displays the increase in Nusselt number as the Reynolds number rises from 100 to 5000. The Nusselt number grows exponentially up to Re = 1000. Beyond Re = 1000, the increase in Nusselt number is directly proportional to Reynolds numbers up to Re = 5000.

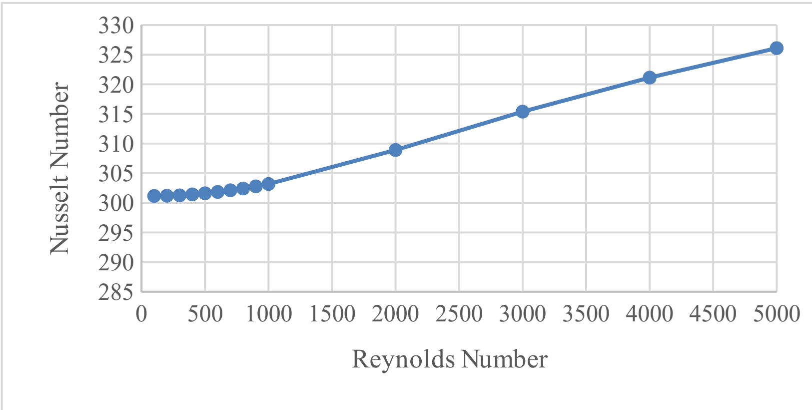

Fig. 6 Reynolds number (from 100 to 5000) vs Nusselt number

It may be stated that in forced convection of a heated cylinder embedded in a Hele-Shaw cell, the Reynolds number does influence the Nusselt number. The Nusselt number increases with the Reynolds number in general.

6. Conclusion

The study focused on the mathematical modelling of forced convection around a heated cylinder enclosed in a Hele-Shaw cell. The governing equations have been derived, solved, and simulated using ANSYS Fluent. The analysis revealed that the Reynolds number significantly impacts the Nusselt number, with exponential growth observed at lower Reynolds numbers up to Re = 1000 and proportional growth at higher Reynolds numbers between Re = 1000 and 5000. The study provides valuable insights into the behaviour of fluid flow and heat transfer in the Hele-Shaw cell system, particularly in the context of a heated cylinder.

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