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An Optimal Control Problem Over Infected Networks

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Abstract - We study an optimal control problem over a network whose nodes are infected by a virus. The infection levels in the network evolve according to a nonlinear dynamical system that belongs to the susceptible-infected-susceptible epidemiological models class. A network designer attempts to regulate the infection levels in the network via adapting the curing rates of the nodes. We show that the optimal controller of this problem exhibits multiple switches between the allowed actions. Further, we present a method for approximating the optimal controller while keeping the deviation from the optimal cost at a minimum. We support our findings by numerical studies.

Keywords: Networks, epidemics, Game Theory, optimal control.

1. Introduction

Different types of infections can spread rapidly over networks via local interactions among the nodes. The modeling and control of the spread of viruses, spam, and rumors in networks have received increased interest from researchers in the last decade (Wang et al. 2003, Van Mieghem et al. 2009, Acemoglu et al. 2013). Besides describing the spread of diseases among humans and animals, epidemiological models can be employed to describe the spread of viruses in mobile and computer networks (Goffman & Newill 1967, Kephart & White 1991, Ganesh et al. 2005).

Various information spread control problems appear in the literature. Shah & Zaman (2011) study the problem of detecting a rumor source in a network. The problem of optimal curing rates allocation was studied by Borgs et al. (2010). A competition to limit influence between two campaigns was modeled and studied by Budak et al. (2011). For a network of nodes running distributed linear averaging, we have studied the interaction between a network designer and an intelligent adversary who compete to control the state of the network (Khanafer et al. 2012, Khanafer et al. 2013).

The main focus of this paper is designing optimal controllers that are capable of reducing the infection levels in the network at minimum cost. To describe the infection diffusion in the network, we adopt the so-called susceptible-infected-susceptible (SIS) n-intertwined Markov model that was recently proposed (Van Mieghem et al. 2009, Van Mieghem & Omic 2013). We allow the network designer to control the curing rates in the network at a predetermined cost. Optimization problems subject to this particular model were previously studied (Preciado et al. 2013, Preciado et al. 2014, Omic et al. 2009); however, these problems were all *static*, and the controllers were not allowed to vary with time. Our main contributions in this paper are as follows:

- We formulate a *dynamic* optimization problem to reduce the infection in the network.
- We present an approximation method that yields a near-optimal controller.

Organization

The rest of this paper is organized as follows. We introduce the SIS n-intertwined Markov model in 2. In Section 3, we introduce the optimal control problem to be solved by the network designer. An approximation method for the optimal controller is presented in Section 4. Numerical studies are provided in Section 5. We conclude the paper in Section 6.

Notation and Mathematical Preliminaries

We denote the all-ones vector by **1**. For notational simplicity, we will often drop the time index. For a matrix $X \in \mathbb{R}^{n\times n}$ with real spectra, the largest eigenvalue is denoted by $\lambda_1(X)$. The operator $\text{diag}(x_1,\ldots,x_n)$ returns a diagonal matrix with $X_{ii} = x_i$, $1 \leq i \leq n$. The absolute value of a real scalar variable is denoted by $|.|$. The ℓ_∞-norm of a vector $x \in \mathbb{R}^n$ is given by $||x||_\infty = \max_{1\leq i\leq n} |x_i|$. Similarly, the ℓ_1-norm of x is given by $||x||_1 = \sum_{i=1}^n |x_i|$.

2. The n-Intertwined Markov Model

In this section, we recall the heterogeneous SIS n-intertwined Markov model (Van Mieghem et al. 2009, Van Mieghem & Omic 2013). Consider a network of n nodes that is described by a connected undirected graph $\mathscr{G} = (\mathscr{V},\mathscr{E})$, where $\mathscr{V}$ is the set of vertices, and $\mathscr{E}$ is the set of edges. An edge in $\mathscr{E}$ between two nodes $i, j \in \mathscr{V}$ is denoted by (i, j). We denote the adjacency matrix of $\mathscr{G}$ by A with entries $a_{ij} \in \mathbb{R}_{\geq 0}$, where $a_{ij} = 0$ if and only if $(i, j) \notin \mathscr{E}$. The proposed model is based on viewing each node in the network as a Markov chain with two states: infected or cured. The curing and infection of each node in the network are described by two independent Poisson processes with rates $u_i > 0$ and $\beta_i > 0$, respectively. The transition rates between the two states depends on the infection probabilities of the neighboring nodes as well as their curing and infection rates. A mean-field approximation is introduced in (Van Mieghem et al. 2009) in order to capture the effect of neighbors on a given node via the total expected infection. This facilitates the derivation of an ODE that describes the evolution of the probability of infection of node i. Let $p_i(t) \in [0,1]$ be the infection probability of node i at time $t \in \mathbb{R}_{\geq 0}$ and define $p = [p_1,\ldots, p_n]^T$. Let $U = \text{diag}(u_1,\ldots,u_n)$, $P = \text{diag}(p_1,\ldots, p_n)$, and $B = \text{diag}(\beta_1,\ldots,\beta_n)$. The n-intertwined Markov model is then given by

$$\dot{p}(t) = (AB-U)p(t) - P(t)ABp(t), \quad p(0) = p_0. \tag{1}$$

A necessary and sufficient condition for stabilizing the state p to the origin is (Preciado et al. 2013, Khanafer et al. 2014):

$$\lambda_1(AB-U) < 0. \tag{2}$$

In an earlier paper (Khanafer et al. 2014), we showed that these dynamics are best-response dynamics of an underlying concave game (Rosen 1965, Başar & Olsder 1999) played by the nodes. The game-theoretic interpretation provides a sufficient condition for stabilizing the origin:

$$\frac{1}{2}\sum_{j\neq i} a_{ij}\beta_j < \delta_i, \quad i = 1,\ldots,n.$$

This condition is obtained via the uniqueness condition for the purse-strategy equilibrium of the underlying game. An important feature of this condition is that it is linear and can be checked in a distributed manner, which makes it appealing for designing distributed algorithms, unlike (2) whose computation requires full information about the network.

3. Optimal Control Problem

We now focus on designing optimal controllers for infected networks. We assume that the designer can control the curing rates u_i of all nodes; however, there is a cost associated with increasing the curing rate of any node. We assume that there are minimum and maximum curing rates such that $\underline{u} \leq u_i(t) \leq \overline{u}$, for all i. The action set of the designer can then be written as

$$U \quad = \quad \{w \in \mathbb{R}^n : \underline{u} \leq w_i \leq \overline{u}\}.$$

The set of admissible controls, $\mathscr{U}$, consists of all functions that are piecewise continuous in time and whose range is U. Given a time interval $[0,T]$, we can formally write

$$\mathscr{U} = \{u : [0,T] \to U \mid u \text{ is a piecewise continuous function of } t\}.$$

The designer aims to reduce the infection probabilities across the network, while minimizing the cost associated with modifying the curing rates. Let $c \in \mathbb{R}^n_{\geq 0}$ be the cost associated with the state, and let $d \in \mathbb{R}^n_{\geq 0}$ be the cost associated with the control. We can then write the cost functional of the designer as follows:

$$J(u) = \int_0^T [c^T p + d^T u]dt.$$

In order to minimize the cost associated with the state, the designer must attempt to stabilize the state to the origin. To this end, we will linearize the dynamics in (1) around the origin to obtain $\dot{p} = (AB - U)p$. Noting that $p_i \sum_{j \neq i} a_{ij}\beta_j p_j \geq 0$, for all i and $p \in [0,1]^n$, we conclude that $(AB - D)p - PABp \leq (AB - U)p$. This serves as a confirmation that the linear part of the dynamics is what is important when the focus is stabilization to the origin. We will therefore work with the linearized dynamics hereinafter.

Consider the following optimal control problem:

$$\begin{array}{ll} \inf\limits_{u \in \mathscr{U}} & J(u) \\ \text{subject to} & \dot{p} = (AB - U)p, \quad p(0) = p_0. \end{array}$$

The Hamiltonian associated with this problem is

$$H(p,q,u) = c^T p + d^T u + q^T (AB - U)p,$$

where q is the costate vector. Assuming an optimal controller exists, the Pontryagin's minimum principle (PMP) (Liberzon 2012) states that there exists a costate vector q satisfying the following conical equations along the optimal trajectory:

$$\dot{p}^\star \;=\; (AB - U^\star)p^\star, \quad p^\star(0) = p_0, \tag{3}$$

$$\dot{q}^\star \;=\; -\frac{\partial}{\partial p} H = -(AB - U^\star)^T q^\star - c, \quad q^\star(T) = 0. \tag{4}$$

Further, the PMP states that the optimal control minimizes the Hamiltonian:

$$u^\star = \arg\min_{\underline{u} \leq u_i \leq \overline{u}} H(p^\star, q^\star, u),$$

which yields the following solution, for $i = 1, \ldots, n$,

$$u_i^\star \;=\; \begin{cases} \overline{u}, & d_i - p_i^\star q_i^\star < 0 \\ \underline{u}, & d_i - p_i^\star q_i^\star > 0 \\ \{\underline{u}, \overline{u}\}, & \text{otherwise} \end{cases} \tag{5}$$

Using the continuity of $q^\star$ and the terminal condition imposed on it, we conclude that $u^\star = \underline{u}\mathbf{1}$ over $[T - \varepsilon, T]$, where $\varepsilon > 0$ is small.

4. An Approximation Result

In this problem, the PMP canonical equations (3)-(5) are intractable. In this section, we propose a method to simplify the controller while not sacrificing optimality by much.

Theorem 1. *Let $u^\star$ be the optimal controller satisfying (5), and let $p^\star$ and $q^\star$ be the optimal state and costate vectors satisfying (3) and (4). Consider a controller $\hat{u}$ such that $\underline{\hat{u}} \leq \hat{u}_i \leq \hat{\overline{u}}$, $i = 1, \ldots, n$. Let $\hat{p}$ and $J(\hat{u})$ be the corresponding state and cost. Then, given an $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\max\left\{\overline{u}, \hat{\overline{u}}\right\} - \min\left\{\underline{u}, \underline{\hat{u}}\right\} < \delta \implies |J(u^\star) - J(\hat{u})| < \varepsilon.$$

Proof. Define $\hat{U} = \mathrm{diag}(\hat{u})$ and $\mu = \max\left\{|\overline{u} - \underline{\hat{u}}|, |\hat{\overline{u}} - \underline{u}|\right\}$. We can then write

$$\begin{aligned} \dot{\hat{p}} &= (AB - \hat{U})\hat{p} \\ &= (AB - U^\star)\hat{p} + (U^\star - \hat{U})\hat{p}, \quad \hat{p}(0) = p_0. \end{aligned}$$

Note that

$$\begin{aligned} ||(U^\star - \hat{U})\hat{p}||_\infty &= \max_{1 \leq i \leq n} |u_i^\star - \hat{u}_i||\hat{p}_i| \\ &\leq \max\left\{\overline{u}, \hat{\overline{u}}\right\} - \min\left\{\underline{u}, \underline{\hat{u}}\right\} = \mu. \end{aligned}$$

Since $\hat{p}_i \leq 1$, it also follows that

$$||u^\star - \hat{u}|| \leq \mu. \tag{6}$$

We also have

$$\begin{aligned} ||(AB-U^\star)p_1 - (AB-U^\star)p_2||_\infty &\leq ||AB-U^\star||_\infty ||p_1-p_2||_\infty \\ &= \left(\max_{1\leq i\leq n} u_i^\star + \sum_{j\neq i} a_{ij}\beta_j\right)||p_1-p_2||_\infty \\ &\leq \left(\overline{u} + \max_{1\leq i\leq n}\sum_{j\neq i} a_{ij}\beta_j\right)||p_1-p_2||_\infty. \end{aligned}$$

Thus, the function $(AB-U^\star)p$ is Lipschitz with constant $L := \overline{u} + \max\limits_{1\leq i\leq n}\sum_{j\neq i} a_{ij}\beta_j$.

The solutions to (3) and (6) can be written as

$$\begin{aligned} p^\star &= p(0) + \int_0^t (AB-U^\star)p^\star ds, \\ \hat{p} &= p(0) + \int_0^t (AB-U^\star)\hat{p} + (U^\star - \hat{U})\hat{p}ds. \end{aligned}$$

We then have

$$\begin{aligned} ||p^\star - \hat{p}||_\infty &\leq \int_0^T ||(AB-U^\star)(p^\star-\hat{p})||_\infty ds + \int_0^t ||(U^\star-\hat{U})\hat{p}||_\infty ds \\ &\leq \mu t + L\int_0^t ||p^\star-\hat{p}||_\infty ds. \end{aligned}$$

Applying the Bellman-Grownwall Lemma to $||p^\star - \hat{p}||_\infty$, we obtain

$$||p^\star - \hat{p}||_\infty \leq \mu t + \mu L \int_0^t s e^{L(t-s)} ds.$$

Integration by parts yields

$$||p^\star - \hat{p}||_\infty \leq \frac{\mu}{L}\left(e^{Lt}-1\right). \tag{7}$$

Now, consider the difference between the optimal cost, and the cost corresponding to $\hat{u}$:

$$\begin{aligned} |J(u^\star) - J(\hat{u})| &\leq \int_0^T |c^T(p^\star-\hat{p})|ds + \int_0^T |d^T(u^\star-\hat{u})|ds \\ &\leq ||c||_1 \int_0^T ||p^\star-\hat{p}||_\infty ds + ||d||_1 \int_0^T ||u^\star-\hat{u}||_\infty ds, \end{aligned}$$

where the second inequality follows by Hölder's Inequality. Using (6) and (7), and the hypothesis that $\mu < \delta$, we can further write

$$\begin{aligned} |J(u^\star) - J(\hat{u})| &\leq \mu\left(\frac{e^{TL}-1-TL^2}{L^2}||c||_1 + T||d||_1\right) \\ &< \delta\left(\frac{e^{TL}-1-TL^2}{L^2}||c||_1 + T||d||_1\right). \end{aligned}$$

The theorem then follows by selecting

$$\delta < \frac{\varepsilon}{\frac{e^{TL}-1-TL^2}{L^2}||c||_1 + T||d||_1}.$$

$\square$

Remark 1. *The costate equation must be solved backward due to its end point condition. This might hinder implementing* $u^\star$*. However, Theorem 1 provides an alternative method, where the sub-optimal cost can be at most* ε *away from* $J(u^\star)$*, without needing to solve the costate equation. For example, assume that* T *is small, and suppose that* $\overline{u} - \underline{u} = 2\delta$*. Then, by choosing* $\hat{u} = \frac{\overline{u}-\underline{u}}{2}\mathbf{1}$ *for all* t*, we guarantee that* $|J(u^\star) - J(\hat{u})| < \varepsilon$*. Hence, by employing this constant strategy, one can avoid all the switching that could be required by* $u^\star$ *without losing much.*

5. Simulation Results

In this section, we demonstrate that the optimal controller (5) can exhibit multiple switches. Consider the network shown in Fig. 1, and let $d = [1,1,10,1,1]^T$ such that node 3 has a high cost on control. Also, let $p(0) = [0.1, 0.01, 0.9, 0.01, 0.01]^T$, where we assigned a high probability of infection to node 3. Let $\underline{u} = 0.1$, $\overline{u} = 1$, $T = 100$, and $c = \mathbf{1}$. Unity infection rates were assigned to all the nodes, i.e, $\beta_i = 1$ for all i. The edge weights a_{ij} were generated randomly.

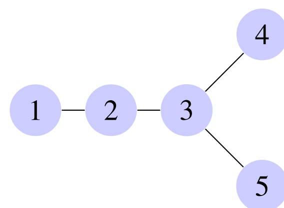

Fig. 1. An infected graph with node 3 having high probability of infection and high cost on control.

Fig. 2 shows the state of the network above after implementing the controller given in (5). Note that $u_3 = \underline{u}$ throughout $[0,T]$, because controlling this node is expensive. Nevertheless, although the neighboring nodes have low initial probability of infections, the optimal controller intelligently increases the curing rates of these nodes, who enjoy low control cost, in order to help cure node 3. It is interesting to note that all the controllers, except u_3, exhibit multiple switches between $\underline{u}$ and $\overline{u}$.

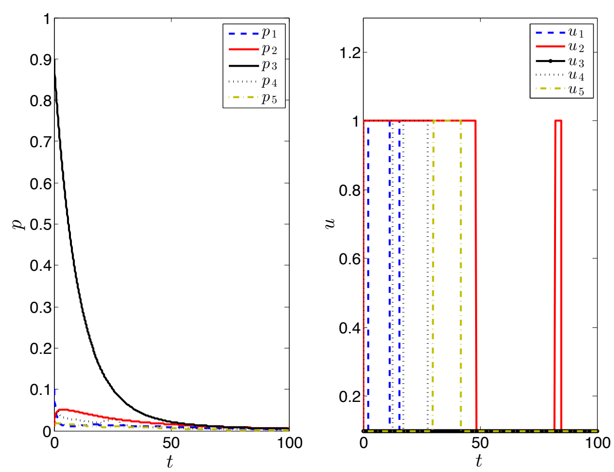

Fig. 2. State and optimal control of a network with a highly infected node whose control cost is high.

6. Conclusion

In this paper, we formulated an optimal control problem that allows the network designer to regulate the infection levels in the network at minimum cost. Using Pontryagin's minimum principle, we derived the optimal controller in terms of the state and costate vectors. We showed that this controller can exhibit multiple switches between the extreme values of the control. Further, we presented an approximation method that allows the designer to use a constant control strategy without compromising optimality much.

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