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Robot Design via Maximization of Fundamental Frequency using Semi-definite Programming

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Abstract - The fundamental frequency of a robotic manipulator typically depends on its configuration. Configurations with the lowest or the highest fundamental frequency are important as they are indicative of the weakest and the strongest configurations in the dynamical sense. This paper describes the use of semi-definite programming (SDP) in finding these two configurations of a robotic manipulator. One key contribution of this work is the computation of the fundamental frequency and its gradient with respect to the configuration variables from the solution of a single SDP. This contribution is used in a search algorithm to identify these two configurations of an exemplary Stewart Platform.

Keywords: Robot design, fundamental frequency, semi-definite programming, Stewart platform.

1. Introduction

A typical goal in robotic manipulator design is a wide operating frequency (Ferretti et al., 1999; Park et al., 2006). This requirement is equivalent to having a high fundamental frequency below which the manipulator can enjoy good dynamical performance. Consequently, approaches that increase the fundamental frequency of mechanisms or structures have been proposed. Examples of such approaches (Zhang et al., 1988; Sivan and Ram, 1996; Lim and Park, 2009; Portman et al., 2000; Menon et al., 2009; Wang et al., 2004) include optimization over stiffness or inertia matrices of robotic arm (Lim and Park, 2009) or parallel mechanisms (Portman et al., 2000; Menon et al., 2009), optimization over placement of support for plates and other structures (Wang et al., 2004), minimum vibration mechanism design (Sivan and Ram, 1996) and many others.

The fundamental frequency of a robotic manipulator is the smallest eigenvalue, λ_{min}, of the corresponding dynamical equations. While the exact dynamical equations can be complex, the determination of λ_{min} involves only the mass and stiffness matrices of the system. Specifically, λ_{min} is the smallest eigenvalue of the generalized eigenvalue problem of the form

$$(\lambda M(q) - K(q))v = 0 \tag{1}$$

where $M(\cdot), K(\cdot) \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ is the generalized coordinate of the mechanism. The design objective is to have as high a value of λ_{min} as possible over some appropriate design variable, x. For example, x is the coordinate of the placement of a support for plate (Wang et al., 2004) while x is the height (h) of Stewart platform (Menon et al., 2009). A typical problem is when x is q. In that case, one is concerned with the configuration of the robotic manipulator that has the lowest fundamental frequency. Clearly, strengthening the manipulator at this configuration means an overall increase in the operating frequency. Optimization of λ_{min} has a long history with many different approaches. Some early approaches optimizes approximations of λ_{min} due to computational considerations. For example, the Rayleigh-Ritz approximation (Pierce and Varga, 1972) is an upper bound of λ_{min} and can be efficiently computed. In the special case where the dependency of the M and K matrices to x is linear or affine, optimization of $\lambda_{min}(x)$ is a convex optimization problem which can be efficiently computed using semi-definite programming (SDP) (Lim and Park, 2009). Unfortunately, the dependence of M and K on x, in the general case, is nonlinear. Many of the convex optimization routines are therefore not suitable for its solution. A direct approach is to consider the nonlinear optimization problem of the form $\min_q \ \lambda_{min}(M^{-1}(q)K(q))$ over the operating range of q. Such an approach has been quite well researched (Nelson, 1976; Mills-Curran, 1988) and expressions of the derivatives of eigenvalues and eigenvectors, including the case of repeated eigenvalues are available. However, it is not clear how the smallest eigenvalue can be efficiently computed. One way is to convert the $\lambda_{min}(M^{-1}(q)K(q))$ into $\lambda_{max}(K^{-1}(q)M(q))$ and solve using power method. However, convergence of power method is known to be slow (Trefethen and Bau, 1997), especially when the two largest eigenvalues are near. Clearly, approaches (most factorization-based approaches) that computes *all* the eigenvalues of $M^{-1}(q)K(q)$ are undesirable. This paper considers the general approach of optimizing $\lambda_{min}(q)$ over q as a nonlinear programming problem (NPP). The algorithmic solution of NPP requires the values of $\lambda_{min}(q)$ and its derivative, $\frac{d\lambda_{min}}{dq}(q)$, at any admissible value of q. This paper shows how these two quantities can be obtained, with minimal additional computations, from a *single* solution of a SDP at a specific choice of q. The presentation is for optimization over q but can be easily adapted for variable x via standard chain rule of differentiation.

The rest of this paper is organized as follows. This section ends with the notations used. Section 2Problem Formulationsection.2 shows the formulation of the eigenvalue maximization problem. Section 3Expressions of λ_{max} and its derivativessection.3 shows the main expressions of λ_{min} and $\frac{d\lambda}{dq}$ from the solution of the SDP. Section 4The Stewart Platform Problemsection.4 shows an example of finding the weakest configuration of a Stewart Platform mechanism within its workspace. Conclusions are drawn in section 5Conclusionssection.5.

The notations used are standard. Matrices and vectors are represented as $A \in \mathbb{R}^{n\times m}$ and $b \in \mathbb{R}^n$ respectively, with A_{ij} and b_k being their corresponding (i, j)-th and k-th elements. Symmetric positive definite matrices, P, is indicated by $P \succ 0$ with $|P|$, $\lambda_{min}(P)$ and $\lambda_{max}(P)$ being its determinant, minimal and the maximal eigenvalues respectively. I_n refers to the identity matrix of order n. A slight abuse of notation is adopted for convenience: $\lambda_{min}, \lambda_{min}(A)$ refer to the smallest eigenvalue in general and that of a specific A matrix respectively while $\lambda_{min}(q) := \lambda_{min}(A(q))$ refers to the smallest eigenvalue as a function of q. The same holds true for λ_{max}. A diagonal matrix is indicated as $diag\{a_{11}, \cdots, a_{nn}\}$ with a_{ii} being the i-th diagonal element. Other notations are introduced when needed.

2. Problem Formulation

The mathematical representations of robotic manipulators can be derived from the Lagrangian approach and has a well-known set of dynamical equations of motion, represented by

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + K(q)q = 0 \tag{2}$$

where $q \in \mathbb{R}^n$ is the vector of standard generalized coordinate, $M(q), K(q) \in \mathbb{R}^{n\times n}$ are the mass and stiffness symmetric matrices respectively and $C(q, \dot{q})$ is the matrix of Coriolis and centrifugal forces.

The natural frequencies of the mechanism is governed by the M and K matrices and the associated eigenvalue problem is that given by (1Introductionequation.1.1). Since M and K are both positive definite, all the eigenvalues of (1Introductionequation.1.1) are real with real eigenvectors. Unfortunately, M and K matrices are generally nonlinear functions of q. A more computationally amiable form is to exploit the positive definiteness of M (or similarly K) and convert (1Introductionequation.1.1) using

$$0 = |\lambda M(q) - K(q)| = |\lambda I_n - M^{-1/2}(q)K(q)M^{-1/2}(q)| := |\lambda I_n - A^{-1}(q)| \tag{3}$$

where $A(q) := M^{1/2}(q)K^{-1}(q)M^{1/2}(q)$ is a symmetric and positive definite matrix. In addition, $\lambda_{min}(A^{-1}(q)) = \frac{1}{\lambda_{max}(A(q))}$, a property that holds for any matrix. This conversion to λ_{max} allows its numerical solution to be obtained as a SDP problem since $w^T(\lambda_{max} I_n - A(q))w = w^T(\lambda_{max} I_n - U\Sigma U^T)w = w^T U(\lambda_{max} I_n - \Sigma)U^T w \geq 0$ for any $w \in \mathbb{R}^n$ where $A = U\Sigma U^T$ is the singular value decomposition of A. Correspondingly, the SDP problem, for a fixed value of q, is

$$\min_t \quad t \tag{4a}$$

$$\text{s.t.} \quad tI_n - A \succeq 0 \tag{4b}$$

With t being the only variable, the optimization problem (4Problem Formulationequation.2.4) is convex and efficient numerical routine exists for its solution. In particular, several codes (Vandenberghe and Boyd, 1996, 1995; Boyd and Vandenberghe, 2004) using the primal-dual interior point method exist and are particularly useful for solving such a problem.

Using the SDP problem of (4Problem Formulationequation.2.4), the overall optimization for finding the weakest configuration becomes

$$\left\{ \begin{array}{ll} \max_q & \lambda_{max}(A(q)) \\ \text{s.t. } \underline{q}_i \leq q_i \leq \bar{q}_i, i = 1, \cdots, n & \end{array} \right. \Leftrightarrow \left\{ \begin{array}{ll} \min_q & -\lambda_{max}(A(q)) \\ \text{s.t. } \underline{q}_i \leq q_i \leq \bar{q}_i, i = 1, \cdots, n & \end{array} \right. \tag{5}$$

where $\underline{q}_i$ and $\bar{q}_i$ are simple operating bounds on q. Hence, each function call of $\lambda_{max}(A(q))$ invokes the SDP problem of (4Problem Formulationequation.2.4) as a subroutine.

Remark 1. *An immediate extension of the above is the problem of finding the strongest configuration, defined by the configuration with the highest fundamental frequency. Using the above development, this problem can be formulated as*

$$\left\{ \begin{array}{ll} \min_q & \lambda_{max}(A(q)) \\ \textit{s.t. } \underline{q}_i \leq q_i \leq \bar{q}_i, i = 1, \cdots, n & \end{array} \right.$$

3. Expressions of λ_{max} and its Derivatives

When the primal-dual interior point algorithm is used for the numerical solution of (4Problem Formulationequation.2.4), a dual variable or the Lagrange variable in the form of a positive semi-definite symmetric matrix, $\Lambda \succeq 0$, is available at the solution of (4Problem Formulationequation.2.4). Suppose the optimal values of t and Λ are t^* and Λ^* respectively. The optimal dual variable satisfies the following two conditions:

$$(t^* I_n - A)\Lambda^* = 0 \tag{6}$$

$$1 - \sum_{i=1}^{n} \Lambda^*_{ii} = 0. \tag{7}$$

The first is the complementary slackness condition of the SDP problem while the second corresponds to the constraint of the dual optimization problem.

The condition of (6Expressions of λ_{max} and its derivativesequation.3.6) is of particular importance. The trivial solution where $\Lambda^* \equiv 0$ is ruled out by condition (7Expressions of λ_{max} and its derivativesequation.3.7). Hence, any non-zero column of Λ^* is an eigenvector corresponding to the smallest eigenvalue t^*. The following two cases are considered:

3.1. Isolated λ_{min}

If t^* is an isolated eigenvalue of A, then $rank(\Lambda) = 1$. Let v be any one of these non-zero columns. It follows from (6Expressions of λ_{max} and its derivativesequation.3.6) that

$$(t^* I_n - A)v = 0 \tag{8}$$

$$\Rightarrow \quad \frac{dt^*}{dq}v - \frac{dA}{dq}v = -(t^* I_n - A)\frac{dv}{dq} \tag{9}$$

Now, pre-multiply v^T to both sides of (9Isolated λ_{min}equation.3.9), then

$$\begin{aligned} \Rightarrow \quad \frac{dt^*}{dq}v^T v - v^T \frac{dA}{dq}v &= -v^T (t^* I_n - A)\frac{dv}{dq} \\ &= -\left[(t^* I_n - A)^T v\right]^T \frac{dv}{dq} \\ &= -\left[(t^* I_n - A)v\right]^T \frac{dv}{dq} = 0 \end{aligned} \tag{10}$$

where the last equation follows from (8Isolated λ_{min}equation.3.8) and the fact that A is symmetric. As a result, we have

$$\Rightarrow \quad \frac{dt^*}{dq} = \frac{v^T \frac{dA}{dq} v}{v^T v} \tag{11}$$

Clearly, $\frac{dt^*}{dq}$ depends on $\frac{dA}{dq}$ whose expression is

$$\frac{dA}{dq} = \frac{dM^{0.5}}{dq}K^{-1}M^{0.5} + M^{0.5}\frac{dK^{-1}}{dq}M^{0.5} + M^{0.5}K^{-1}\frac{dM^{0.5}}{dq}. \tag{12}$$

The expressions of $\frac{dM^{0.5}}{dq}$ and $\frac{dK^{-1}}{dq}$ are typically not easily available from matrices $M(q)$ and $K(q)$. Instead only $\frac{dK}{dq}$ and $\frac{dM}{dq}$ are available from $M(q)$ and $K(q)$. Hence, additional steps are needed in the form of

$$\frac{dK^{-1}}{dq} = -K^{-1}\frac{dK}{dq}K^{-1} \tag{13}$$

$$\frac{dM^{0.5}}{dq}M^{0.5} + M^{0.5}\frac{dM^{0.5}}{dq} = \frac{dM}{dq} \tag{14}$$

The last equation can be rewritten as a linear equation with unknown variables corresponding to the elements of $\frac{dM^{0.5}}{dq}$ and can be solved with the knowledge of $\frac{dM}{dq}$. In fact, it can be shown that the solution obtained is unique, see *Remark* 2rem.2. Similarly, $\frac{dK^{-1}}{dq}$ is obtained from (13Isolated λ_{min}equation.3.13) knowing $\frac{dK}{dq}$.

Remark 2. *Equation (14Isolated λ_{min}equation.3.14) can be rewritten as $\bar{A}\bar{x} = \bar{b}$ for some appropriate $\bar{A} \in R^{n^2 \times n^2}$ and $\bar{b}$ with x being the n^2 elements of $\frac{dM^{0.5}}{dq}$, also known as the Lyapunov equation. The eigenvalues of $\bar{A}$ are $(\mu_i + \mu_j)$ where μ_i and μ_j are any two eigenvalues of $M^{0.5}$. Since $M^{0.5}$ is positive definite, none of its eigenvalues is zero and this implies that $\bar{A}$ has no zero eigenvalue, leading to the uniqueness of the solution of $\frac{dM^{0.5}}{dq}$.*

In many applications, it may also be useful to know the derivatives of eigenvector at λ_{max}. The remaining of this subsection shows the procedure for doing so. A few well-known properties are first stated. The eigenvector of A corresponding to $\lambda_{max}(A)$ is also the eigenvector of $\lambda_{min}(A^{-1})$. This follows because $(\lambda_{max} I_n - A)v = A(A^{-1} - \frac{1}{\lambda_{max}} I_n)v = 0$. Furthermore, when v is the eigenvector of $M^{-1/2}KM^{-1/2}$, $w := M^{0.5}v$ is the eigenvector of $M^{-1}K$ for the same eigenvalue. Hence, the derivative of eigenvector of $M^{-1}K$ is

$$\frac{dw}{dq} = M^{0.5}\frac{dv}{dq} + \frac{dM^{0.5}}{dq}v$$

where $\frac{dM^{0.5}}{dq}$ is that given by (14Isolated λ_{min}equation.3.14). From (9Isolated λ_{min}equation.3.9) and the fact that $(t^*I - A)$ is singular, it follows that

$$\frac{dv}{dq} = \bar{y} + \alpha y_n$$

where $\bar{y} \in \mathbb{R}^n$ is a basic solution and $y_n \in \mathbb{R}^n$ is any null space vector of $(t^*I_n - A)$ and α is any real constant. More exactly, let $Y\Sigma_A Y^T$ be the singular value decomposition of $(t^*I_n - A)$ with $Y = [y_1\ y_2 \cdots y_n]$ and $\Sigma_A = diag\{\sigma_1, \sigma_2, \cdots, \sigma_{n-1}, 0\}$. Then, $\bar{y} = Y_{n-1}\Sigma_{n-1}^{-1}Y_{n-1}^T(dA/dq - dt^*/dq)v$ where $Y_{n-1} = [y_1\ y_2 \cdots y_{n-1}]$ and $\Sigma_{n-1}^{-1} := diag\{\sigma_1^{-1}, \sigma_2^{-1}, \cdots, \sigma_{n-1}^{-1}\}$. The choice of α can be resolved to achieve a unique $\frac{dv}{dq}$. Since eigenvector corresponding to an eigenvalue is only unique up to a constant multiple, a useful choice of normalization of eigenvector is to chose $max_i|v_i|$ = 1. Let $i^* = arg\ \ max_i|v_i|$. The constant α can be obtained such that $\frac{dv_{i^*}}{dq} = 0$.

3.2. Non-isolated λ_{min}

Due to the symmetry in many mechanism design, the possibility of having repeated eigenvalues corresponding to $\lambda_{max}(A)$ exists. Suppose $\lambda_{max}(A)$ is repeated r times. This means that the $rank(\Lambda) \leq r$. The more likely situation is that $rank(\Lambda) = r$ and Λ contains r linearly independent columns that spans the eigensubspace corresponding to $\lambda_{max}(A)$. It is also possible that $rank(\Lambda) < r$ although such cases are rare. The details of obtaining the expressions of the derivatives of repeated eigenvalues and eigenvectors will not be covered here since they have been presented in (Nelson, 1976; Mills-Curran, 1988; Friswell, 1996; Andrew and Tan, 1998) and the connections made above on the solution of the SDP problem.

4. The Stewart Platform Problem

This section shows derivations of the M and K matrices of a Stewart Platform mechanism but with no intention to develop the full dynamical equations. The derivation is also brief as these expressions are quite well known (Portman et al., 2000; Menon et al., 2009). The M and K expressions are then used in (5Problem Formulationequation.2.5) to identify the weakest configuration in its workspace.

Stewart Platform (SP) is a parallel mechanism well-known for accomplishing tasks that require high-precision, high load carrying capacity and good dynamic performance in three-dimensional space (Merlet, 2006; Portman et al., 2000; Menon et al., 2009). As shown in Fig. **??**, the Stewart Platform has two rigid bodies, the base and the platform, connected through six extensible legs. Two reference frames are needed:

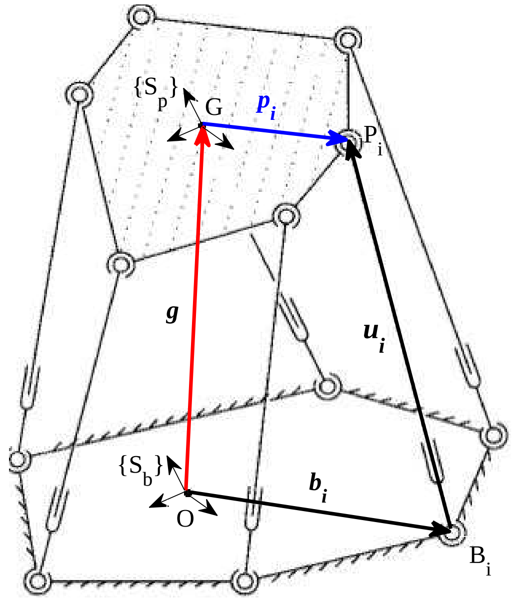

Fig. 1. The Stewart Platform

S_b is a fixed frame centered at O and attached to the base while S_p is a moving frame with origin G fixed at the center of mass of the platform with axes aligned with its principal axes.

The generalized coordinate q is $\{x,y,z,\alpha,\beta,\gamma\}$ where (x,y,z) defines the position of G from O under frame S_b, indicated by the vector g in the figure and (α,β,γ) is the z-y-x Euler angles that defines the orientation of S_p with respect to S_b. Using the coordinate systems set up, the vector that defines the i-th leg is

$$u_i(q) = (\vec{OP_i} - \vec{OB_i}) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + R(\alpha,\beta,\gamma)p_i - b_i \tag{15}$$

where p_i and b_i are the vectors from G to P_i and O to B_i respectively (as shown in figure) and $R(\alpha,\beta,\gamma) := R_z(\alpha)\cdot R_y(\beta)\cdot R_x(\gamma)$ is the product of the rotation matrices along z,y and x axes by the respective Euler angles.

Like most dynamical analysis (Portman et al., 2000; Menon et al., 2009) of Stewart Platform, the following assumptions are made: the platform is a rigid body while the legs have negligible mass with small deflections in the elastic region.

The inertial matrix of the platform can be shown to be (Menon et al., 2009)

$$M = diag(m,m,m,I_1,I_2,I_3) \tag{16}$$

where m is the mass and I_1, I_2, I_3 are the moments of inertia with respect to the principal axes of the platform respectively. Other off-diagonal terms exist when the full dynamical equations are considered but they are not elements of the inertial matrix. Clearly, M does not depend on q and $\frac{\partial M}{\partial q} = 0$.

To obtain expression of the stiffness matrix, the relationship between the length of the leg, ℓ_i and q is needed. From (15The Stewart Platform Problemequation.4.15),

$$\ell_i^2(q) = u_i^T(q)u_i(q) \tag{17}$$

since ℓ_i is the length of u_i. Correspondingly, the velocities of the length, $\dot{\ell}$, and the generalized coordinates, $\dot{q}$, are related through the well-known Jacobian matrix relation in the form of

$$\dot{\ell} = J(q)\dot{q} \tag{18}$$

where $\dot{\ell} = [\dot{\ell}_1, \cdots, \dot{\ell}_6]^T$ and $\dot{q} = [\dot{x}, \dot{y}, \dot{z}, \omega_x, \omega_y, \omega_z]^T$. The procedure of getting the expression of J is quite well-known (Merlet, 2006). It is obtained by taking derivatives of (17The Stewart Platform Problemequation.4.17) together with (15The Stewart Platform Problemequation.4.15). Following it, the expression is

$$J(q) = \begin{bmatrix} \hat{u}_1^T & s_1^T \\ \vdots & \vdots \\ \hat{u}_6^T & s_6^T \end{bmatrix} \tag{19}$$

where

$$\hat{u}_i := u_i/\ell_i, \tag{20}$$

$$s_i := \vec{OP_i} \times \hat{u}_i \tag{21}$$

with s_i being the moment of $\hat{u}_i$ relative to O. It is also well-known from (18The Stewart Platform Problemequation.4.18) that

$$\delta\ell = J(q)\,\delta q. \tag{22}$$

In addition, the relation between the forces acting on the platform by the legs, $f = [f_1, \cdots, f_6]^T$, and the components of the generalized forces (Γ) is

$$\Gamma = J^T(q)f \tag{23}$$

The legs are commonly modeled as axial springs. Hence, the force of the i^{th} leg, f_i, is proportional to its elastic deformation $\delta\ell_i$ or $f_i = k_i\,\delta\ell_i$, where $k_i = \frac{EA}{\ell_i}$ is the stiffness of each leg and E, A are the Youngs modulus and cross-sectional area of the leg. In matrix form,

$$f = D\,\delta\ell \tag{24}$$

where, $D = diag(k_1, \cdots, k_6)$. From (22The Stewart Platform Problemequation.4.22),(24The Stewart Platform Problemequation.4.24) and (23The Stewart Platform Problemequation.4.23), it follows that $\Gamma = (J^T(q)DJ(q))\,\delta q := K(q)\delta q$. Hence, stiffness matrix is

$$K(q) = J^T(q)DJ(q). \tag{25}$$

Taking the derivatives of the above with respect to q_k,

$$\frac{\partial K(q)}{\partial q_k} = \frac{\partial J^T(q)}{\partial q_k}DJ(q) + J^T(q)D\frac{\partial J(q)}{\partial q_k}, \tag{26}$$

$$\text{with } \frac{\partial J}{\partial q_k} = \begin{bmatrix} \frac{\partial \hat{u}_1^T}{\partial q_k} & \frac{\partial s_1^T}{\partial q_k} \\ \vdots & \vdots \\ \frac{\partial \hat{u}_6^T}{\partial q_k} & \frac{\partial s_6^T}{\partial q_k} \end{bmatrix} \tag{27}$$

where

$$\frac{\partial \hat{u}_i}{\partial q_k} = \frac{\ell_i \frac{\partial u_i}{\partial q_k} - \frac{\partial \ell_i}{\partial q_k} u_i}{\ell_i^2}. \tag{28}$$

The expression of $\frac{\partial u_i(q)}{\partial q_k}$ can be computed directly from (15The Stewart Platform Problemequation.4.15) while that of $\frac{\partial \ell_i}{\partial q_k}$ follows from differentiating (17The Stewart Platform Problemequation.4.17) which leads to

$$\frac{\partial \ell_i}{\partial q_k} = \frac{1}{2}\left(\hat{u}_i^T(q) \frac{\partial u_i(q)}{\partial q_k} + \frac{\partial u_i(q)}{\partial q_k}^T \hat{u}_i \right)$$

Suppose $\vec{OP_i} = (r_{x_i}, r_{y_i}, r_{z_i})^T$. The expression of s_i of (21The Stewart Platform Problemequation.4.21) can be rewritten, using standard equivalence for cross product of two vectors, as

$$s_i = \vec{OP_i} \times \hat{u}_i \equiv [\vec{OP_i}]\hat{u}_i$$

where $[\vec{OP_i}] = \begin{bmatrix} 0 & -r_{z_i} & r_{y_i} \\ r_{z_i} & 0 & -r_{x_i} \\ -r_{y_i} & r_{x_i} & 0 \end{bmatrix}$. With this,

$$\frac{\partial s_i}{\partial q_k} = \frac{\partial [\vec{OP_i}]}{\partial q_k} \hat{u}_i + [\vec{OP_i}] \frac{\partial \hat{u}_i}{\partial q_k} \tag{29}$$

and equations $\frac{\partial K}{\partial q}$ can now be computed from (26The Stewart Platform Problemequation.4.26)-(29The Stewart Platform Problemequation.4.29).

As a summary, expressions of $\frac{-\partial \lambda_{max}(q)}{\partial q}$ and $\frac{\partial A(q)}{\partial q}$ are needed for the NPP problem of (5Problem Formulationequation.2.5). Suppose the SDP primal and dual solutions at a given q are (t^*, Λ^*). Then, $-\lambda_{max}(q) = -t^*$ with gradient $\frac{-\partial \lambda_{max}(q)}{\partial q} = -\frac{v^{*T} \frac{\partial A(q)}{\partial q} v^*}{v^{*T} v^*}$, where v^* is any column of the Λ^* and $\frac{\partial A(q)}{\partial q}$ is computed from (12Isolated λ_{min}equation.3.12)-(14Isolated λ_{min}equation.3.14) and (26The Stewart Platform Problemequation.4.26)-(29The Stewart Platform Problemequation.4.29).

4.1. Numerical Results

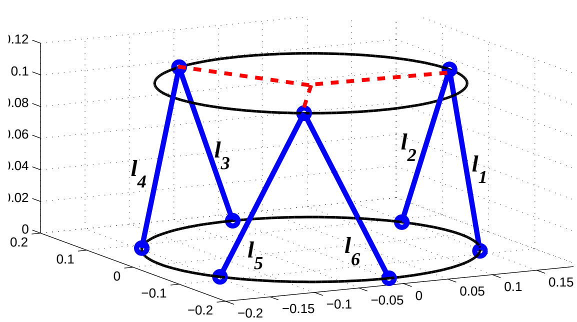

Fig. 2. Scheme for the calculation of the SP natural frequencies.

The parameters of the Stewart Platform used in this numerical study are taken from (Portman et al., 2000) and the platform is shown in Fig. 2Scheme for the calculation of the SP natural frequencies.figure.caption.2. The centers of the joints P_i and B_i lie on circles with the radius of $r_p = 0.156\,m$ and $r_b = 0.169\,m$, respectively. The arrangement of the six legs on the base is axisymmetric; the ends of any two consecutive legs are separated by 60°. The other end of the legs are arranged to meet platform at 120° apart. The axial stiffness constants of all legs are set to be $k_i = 10^6 N/mm$ and the height of the platform at its neutral position from the base, $\bar{h}$, is $0.105\,m$. The rest of the parameters are $m = 25.32\,kg$, $I_x = I_y = 0.4167$ and $I_z = 0.2598\,kg\,m^2$. The range of admissible q for which the stiffness model is valid is $Q := \{(x,y,z,\alpha,\beta,\gamma) : |x| \leq 0.05, |y| \leq 0.05, |z-\bar{h}| \leq 0.01, |\alpha| \leq \frac{5\pi}{180}, |\beta| \leq \frac{5\pi}{180}, |\gamma| \leq \frac{5\pi}{180}\}$.

The nonlinear optimization (5Problem Formulationequation.2.5) uses the BFGS algorithm (Kelley, 1999; Byrd et al., 1995; Nocedal and Wright, 2006) with box constraints for its solution. BFGS is a quasi-Newton method, in which the Hessian matrix of second derivatives need not be evaluated directly. Instead, the Hessian matrix is approximated using rank-one updates specified by previous gradient evaluations.

The SP considered here is symmetrical with respect to 3 planes, each of them containing the z-axis of $\{S_b\}$ and one of the three dashed-lines shown in Fig. 2Scheme for the calculation of the SP natural frequencies.figure.caption.2. Hence, the search for the weakest configuration can be restricted to $\bar{Q} := \{(x,y,z,\alpha,\beta,\gamma) : x \leq 0.05, -0.05 \leq y \leq 0, x - \frac{1}{\sqrt{3}}y \geq 0, |z-\bar{h}| \leq 0.01, |\alpha| \leq \frac{5\pi}{180}, |\beta| \leq \frac{5\pi}{180}, |\gamma| \leq \frac{5\pi}{180}\}$. Any given configuration in this region means that 5 other configurations exist in Q, each obtained by the image of the given configuration reflected about one of the 3 planes of symmetry. Hence, multiple local minima exist in Q due to the symmetry.

There is also a possibility that local minima exist within $\bar{Q}$ not due to symmetry. Our approach to mitigate this effect is to invoke the BFGS algorithm with multiple initial configurations. These initial configurations correspond to points of a uniform grid taken over $\bar{Q}$. Clearly, probability of finding the global optimal increases with the number of initial points. The results of this strategy on SP are presented in Table 1Computational results for minimizing λ_{min} table.caption.3. These results are obtained using the "cvx" optimization routine on the Matlab 7 platform and the projected-BFGS package (Kelley, 2011). The computations are performed on a dual-core Macbook Pro with 3.2 GHz processor and 4 GB of memory.

Table 1. Computational results for minimizing λ_{min}

No. of initial points	λ_{min} $(\times 10^3)$	No. of distinct local minima	Wall time (sec.)
10	12.3457	5	5.253
20	12.2470	8	8.373
30	11.7036	12	12.740
50	11.3308	18	21.201
100	11.3308	24	45.840
500	11.3308	30	225.357

Remark 3. *The BFGS algorithm terminates under two situations: (i) a configuration q in the interior of $\bar{Q}$ with $\frac{d\lambda_{max}(q)}{dq} = 0$; (ii) q is at the boundary of $\bar{Q}$ while satisfying the Karush-Kuhn-Tucker optimality conditions.* All *the solutions shown in Table 1Computational results for minimizing λ_{min} table.caption.3 are of type (ii). This fact also suggests that the optimal solution is a result of the constraints - further deterioration of the fundamental frequency results if the constraints are relaxed by enlarging Q.*

In summary, the weakest configuration found in $\bar{Q}$ is

$$q^* = (0.05, -0.05, (\bar{h} - 0.01), 0.0873, -0.0873, -0.0873),$$

shown in Fig. 3Optimal configuration q^*figure.caption.4, for which $\lambda_{min}(A^{-1}(q^*)) = \frac{1}{t^*} = 11.3308 \times 10^3\ rad^2/s^2$. It is possible to compute, albeit with some careful manipulations, the other 5 configurations obtained from q^* by reflecting about the 3 symmetrical planes. These 5 configurations are

$$q_2 = \begin{bmatrix} 0.05 \\ 0.05 \\ \bar{h}-0.01 \\ -0.0873 \\ -0.0873 \\ 0.0873 \end{bmatrix}, q_3 = \begin{bmatrix} -0.0183 \\ 0.0683 \\ \bar{h}-0.01 \\ 0.0010 \\ 0.1915 \\ 0.0965 \end{bmatrix}, q_4 = \begin{bmatrix} -0.0183 \\ -0.0683 \\ \bar{h}-0.01 \\ -0.0011 \\ 0.1915 \\ -0.0965 \end{bmatrix}, q_5 = \begin{bmatrix} -0.0683 \\ 0.0183 \\ \bar{h}-0.01 \\ 0.0485 \\ 0.0745 \\ 0.0142 \end{bmatrix}, q_6 = \begin{bmatrix} -0.0683 \\ -0.0183 \\ \bar{h}-0.01 \\ -0.0485 \\ 0.0745 \\ -0.0142 \end{bmatrix}.$$

It can be verified that only q_2 is within Q and satisfies condition (ii) of *Remark* 3rem.3. The other configurations, $q_3, \cdots, q_6$ are all outside of Q and does not constitute a valid optimal solution of (5Problem Formulationequation.2.5).

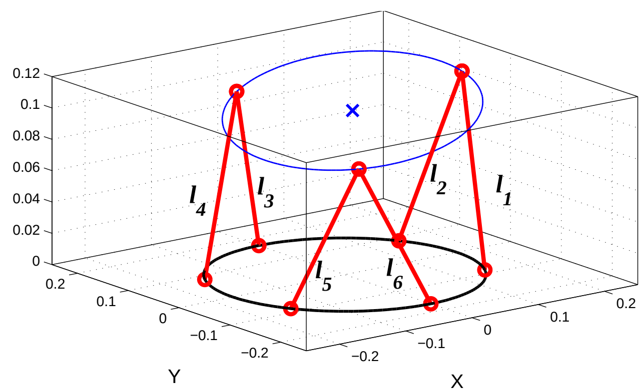

Fig. 3. Optimal configuration q^*

As mentioned in *Remark* 1rem.1, it is also possible to find the strongest configuration of the SP that has the maximal fundamental frequency. The numerical results for the computation of this configuration is presented in Table 2Computational results for maximizing λ_{min}table.caption.5. Again by increasing the number of initial points, the solution obtained is $\bar{q} = (0, 0, \bar{h}-0.01, 0, 0, 0)$.

Table 2. Computational results for maximizing λ_{min}

No. of initial points	λ_{min} $(\times 10^3)$	No. of distinct local minima	Wall time (sec.)
10	39.3281	6	5.230
20	39.4830	10	9.098
30	39.6752	16	17.0192
50	39.7496	26	25.0484
100	39.8461	35	54.1743
500	39.8461	44	235.457

The approach provided here is likely to be more efficient than those using multiple grid search methods or (gradient-free) stochastic search algorithms (see for example (Menon et al., 2009)). The gradient evaluation described in Section 3Expressions of λ_{max} and its derivativessection.3, speeds up the search algorithm significantly resulting in fast convergence of the algorithm. In the above example, the average time needed for the convergence of the algorithm to a local minima is less than half a second (wall time).

5. Conclusions

This paper describes the use of the semi-definite programming (SDP) in finding the weakest and the strongest configurations of a mechanism. Computation of the fundamental frequency at any configuration and its gradient with respect to the configuration variables obtained from a *single* SDP solution are discussed. These are used in a BFGS algorithm to find the weakest and strongest configurations of a Stewart Platform.

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