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Leader-Follower Consensus Modeling Representative Democracy

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Abstract- In this work, we study a leader-follower consensus protocol where both leaders and followers negotiate their states over a stochastically-switching network. The model incorporates the phenomenon of numerosity, which limits the perception of exact numbers. We derive a closed form expression for the asymptotic convergence factor that provides a necessary and sufficient condition for convergence and is used to study the expected decay rate of disagreement among agents. These results are validated with Monte Carlo simulations and we explore the dependence of the asymptotic convergence factor on model parameters using numerical simulations. This system can be used to model decision making in a representative democracy, where representatives negotiate among themselves and drive the opinion of the population.

Keywords: Communication networks, Consensus, Leader follower, Stochastic systems

1 Introduction

Collective decision making can be modeled as a consensus protocol, defined as an algorithm for a multi-agent system with an equilibrium when all agents hold a common state. The wide range of engineering and science applications for consensus protocols, such as unmanned aerial vehicles (Beard et al. 2002) and autonomous underwater vehicles (AUVs) (Maczka et al. 2009), is supported by a large theoretical literature exploring these problems (Ren & Beard 2007, Abaid & Porfiri 2011, Pereira 2010, Abaid & Porfiri 2012). Within this literature, conditions to reach consensus have been studied varying the underlying network of agent communication. Networks may be static (Sipahi & Acar 2008) or switching, and the latter case may be further divided into those updating from a deterministic sequence (Ren & Beard 2007) or from realizations of a random variable (Abaid & Porfiri 2011). Within the consensus literature, closed form results for consensus conditions and convergence speed are limited to a small set of known topologies, for example, Erdos-Renyi random networks in (Pereira 2010) and numerosity-constrained (NC) networks in (Abaid & Porfiri 2011, 2012).

In social systems, a variety of constraints restricts the communication between individuals from all-to-all. Among these constraints, the perception of numbers impacts many social species, including fish (Tegeder & Krause 1995) and humans (Piazza & Izard 2009). The so-called numerosity constraint, defined in (Piazza & Izard 2009), limits the perception of exact numbers across species. Another striking feature of social groups is leadership by an individual or subset of the group, studied for example in fish schools (Couzin et al. 2011). This group behavior may be modeled as leader-follower consensus, which partitions the agents into two types: leaders and followers (Abaid & Porfiri 2012, Xiaohong & Qinghe 2013). In general, the leaders have access to more information and attempt to drive the entire system to a desired common state through their updating protocol. We comment that the leaders whose states are time-variant and update dynamically according to a consensus protocol are rarely considered in the literature (Song et al. 2013) and closed form expressions relating the consensus among leaders to that of the whole system are particularly absent.

In this paper, we study a leader-follower consensus protocol over a stochastically-switching network to model the scenario, where individuals in social groups share information with a dynamic and stochastic set of peers, inspired by the numerosity constraint in biological systems. In (Abaid & Porfiri 2012), the authors studied a similar model, where the leaders have a common state which is constant with time. In this work, the leaders states are time-variant and update as they interact with other leaders, while the followers interact with any other agent irrespective of leader or follower. We define the dynamic leaders to be more influential than followers as they only negotiate with a subset of the total group, while their states propagate through the population via the directed interactions they may have with the followers. This problem models a representative democracy, where policies are decided upon by a subset of representatives whose decisions are disseminated to the entire populace (Mezey 2008). This idealized system incorporates the ultimate goal of democracy, in that consensus among representatives is required for making public policy and consensus among all agents represents a populace whose opinions align with its representatives. In addition, we incorporate the numerosity constraint, which is known to impact decision making across social species.

Here, we study such a system with dynamic leaders and establish necessary and sufficient conditions for consensus in terms of the mean square stability of the disagreement among agents. A closed form expression for the asymptotic convergence factor, which measures the rate of convergence to consensus, is established. It is important to note that, in (Abaid & Porfiri 2012), the state of consensus of the entire system was fixed by the set leaders' common state, whereas in this work the leaders' states start from random initial conditions and converge over time.

2 Problem Statement

We consider a system having N agents, with l agents serving as leaders and f agents as followers, where $f, l \geq 3$ and $l + f = N$. The sets $\mathscr{F} = \{1, 2, \cdots, f\}$, $\mathscr{L} = \{f+1, f+2, \cdots, N\}$, and $\mathscr{N} = \mathscr{F} \cup \mathscr{L}$ are used to denote the indices of followers, leaders, and total number of agents, respectively. The agents communicate over a stochastically-switching directed network which is numerosity constrained (Abaid & Porfiri 2011) during discrete time steps. At each time step, agents communicate with n randomly selected neighbors where $n \in \{1, \ldots, \min\{f-1, l-1\}\}$ is constant over all time steps and agents in the system. The followers are assumed to communicate with n neighbors which are selected from both leaders and followers, whereas the leaders communicate with n other leaders.

The communication network, at each time step $k \in \mathbb{Z}^+$, is defined through the graph Laplacian $L_k \in \mathbb{R}^{N \times N}$. The first f rows of L_k i.e. row $i \in \mathscr{F}$, define the followers' interaction graph, where each row represents how a follower is connected to the rest of the $(N-1)$ agents in the system. The last l rows of L_k represent the leaders' interaction graph, i.e. when $i \in \mathscr{L}$, and show the connections are restricted to the leaders. Each row of L_k has a diagonal entry equal to n and off-diagonal entries comprising n "-1's" and $N-n-1$ "0's", by definition. Specifically, when $i \in \mathscr{F}$, -1 can appear along the $N-1$ off-diagonal positions with equal probability, whereas when $i \in \mathscr{L}$, the first f columns are 0's and the appearance of -1's along the remaining $l-1$ off-diagonal positions is equally likely. In other words, the first f columns in the leaders' sub-system has all the elements equal to zero, since leaders do not receive information from the followers. Due to unidirectional communication among the agents, L_k is not necessarily symmetric, but it has zero row sum. Thus, $L_k 1_N = 0_N$, where the vector $1_N \in \mathbb{R}^{N \times 1}$ have all entries equal to 1 and the vector $0_N \in \mathbb{R}^{N \times 1}$ have all entries equal to zero.

At time step k, the agents' states are given by the vector $x_k = [x_k^{f^T} \; x_k^{l^T}]^T \in \mathbb{R}^{N \times 1}$, where $x_k^f \in \mathbb{R}^{f \times 1}$ represents the state vector for the followers and $x_k^l \in \mathbb{R}^{l \times 1}$ represents the state vector for the leaders. The state vector of the entire system is updated according to the discrete-time consensus protocol

$$x_{k+1} = (I_N - \mathscr{E} L_k)x_k, \tag{1}$$

where I_N is an identity matrix of size N and x_0 is a random initial condition. The diagonal matrix $\mathscr{E} \in \mathbb{R}^{N\times N}$ consists of the constant diagonal entries $\varepsilon > 0$. The parameter ε, also called persuasibility, acts as a weighting parameter and determines how the agents update their states using the information received from the neighbors at each time step. We say that the system reaches consensus when agents attain common state variable, $x = s1_N$, where $s \in \mathbb{R}$.

3 Analysis

In this section, we define and derive the closed form expression for the asymptotic convergence factor for the consensus protocol in (1). This quantity provides a necessary and sufficient condition for consensus and captures the rate of convergence to consensus.

3.1 Preliminary results

Considering a discrete-time linear system, we write

$$x_{k+1} = W_k x_k, \tag{2}$$

where $W_k \in \mathbb{R}^{N\times N}$ are independent, identically distributed random matrices. For (2) to be a consensus protocol, W_k must have the property $W_k 1_N = 1_N$, that is, elements in $\mathrm{span}(1_N)$ are equilibria of the system.

Following (Abaid & Porfiri 2011), we project the consensus problem (1) on the disagreement space, in terms of a disagreement variable ξ_k defined as $\xi_k = Q^T x_k \in \mathbb{R}^{N-1}$. The matrix $Q \in \mathbb{R}^{N\times(N-1)}$ has the properties $Q^T 1_N = 0_N$, $Q^T Q = I_{N-1}$, and $QQ^T = R$ where $R = I_N - 1_N 1_N{}^T$. Thus, the disagreement dynamics is given by the relation, $\xi_{k+1} = \tilde{W}_k \xi_k$, where $\tilde{W}_k = Q^T W_k Q \in \mathbb{R}^{(N-1)\times(N-1)}$.

Following (Zhou & Wang 2009), the asymptotic convergence factor can be written in terms of the disagreement dynamics as

$$r_a = \sup_{\|\xi_0\|\neq 0} \lim_{k\to\infty} \left(\frac{\mathbf{E}\left[\|\xi_k\|^2\right]}{\|\xi_0\|^2} \right)^{1/k} \tag{3}$$

where $\mathbf{E}[\cdot]$ is the expected value. In (Abaid & Porfiri 2011), it is shown that the asymptotic convergence factor is less than one if and only if the disagreement system is mean square stable, that is, if the system in (2) is mean square consentable.

We define, using the same notation of (Abaid & Porfiri 2011),

$$G = (R\otimes R)(I_{N^2} - \varepsilon(\mathbf{E}[L]\oplus\mathbf{E}[L]) + \varepsilon^2 \mathbf{E}[L\otimes L]) \tag{4}$$

where $\otimes$ and $\oplus$ denote Kronecker product and Kronecker sum, respectively. From (Abaid & Porfiri 2011), we know that the asymptotic convergence factor is equal to the spectral radius of G.

Thus we have the following proposition for assessing the consentability of the system:

Proposition 1. *The system* (1) *is consentable in the mean square sense if and only if*

$$r_a = \rho(G) < 1. \tag{5}$$

Proof. The proof of the proposition can be found in (Abaid & Porfiri 2011). □

In the next subsection, we compute G by a counting technique. Then, we write a closed form expression for the asymptotic convergence factor for the consensus protocol in (1) by calculating the eigenvalues of G and associated eigenvectors and applying Proposition 1.

3.2 Main results

The set of all possible distinct Laplacian matrices is denoted as $\hat{L} = \{L^{(1)}, L^{(2)}, \cdots, L^{(p)}\}$, where p corresponds to the total number of unique realizations of the Laplacian matrices for a given set of parameters l, f, and n. Following similar steps in (Abaid & Porfiri 2011, 2012, Porfiri 2011, 2012), we calculate $\mathbf{E}[L]$ and $\mathbf{E}[L\otimes L]$ using a counting technique assuming that the appearance of each of these p matrices is equally likely. The matrix $\mathbf{E}[L]$ has diagonal components equal to n and and the off-diagonal $ij-$th components, where $i \neq j$, are given by $-n/(N-1)$ when $i \in \mathscr{F}$ and $j \in \mathscr{N}$; $-n/(l-1)$ when $i \in \mathscr{L}$ and $j \in \mathscr{L}$; and 0 when $i \in \mathscr{L}$ and $j \in \mathscr{F}$. Therefore we can write

$$\mathbf{E}[L] = \frac{nN}{N-1}R_f + \frac{nl}{l-1}R_l, \tag{6}$$

where $R_f = \sum\limits_{q=1}^{f} e_q e_q^T - \frac{1}{N}1_f 1_N^T$ and $R_l = \sum\limits_{q=f+1}^{N} e_q e_q^T - \frac{1}{l}1_l 1_l^T$, the vector $e_q \in \mathbb{R}^{N\times 1}$ has 1 in the qth row and zeros in the remaining rows, and $1_f = \sum_{q=1}^{f} e_q$, and $1_l = \sum_{q=f+1}^{N} e_q$.

To compute $\mathbf{E}[L\otimes L]$, we note that $L_{ij} \in \{0, -1, n\}$, and $L\otimes L$ has terms of the form $L_{ij}L_{km}$ from the definition of Kronecker product. Therefore, $L\otimes L$ can have values $\{0, 1, -n, n^2\}$. To calculate the distinct values for elements of $\mathbf{E}[L\otimes L]$, we consider the six possible cases for the indices i, j, k, and m, that is, 1) $i = j$, $k = m$; 2) $i = j, k \neq m$; 3) $i \neq j, k = m$; 4) $i \neq j, k \neq m, i = k, j = m$; 5) $i \neq j, k \neq m, i = k, j \neq m$; and 6) $i \neq j, k \neq m, i \neq k$. In general, the diagonal blocks of $\mathbf{E}[L\otimes L]$ have the form

$$\mathbf{E}[L\otimes L]_{ii} = \frac{n^2 N}{N-1}R_f + \frac{n^2 l}{l-1}R_l \text{ when } i \in \mathscr{N}, \tag{7}$$

and the off-diagonal blocks can have three different forms, depending on the values of i and j,

$$\mathbf{E}[L\otimes L]_{ij} = \frac{-n^2 N}{(N-1)^2}R_f - \frac{n^2 l}{(N-1)(l-1)}R_l + \Delta_1 e_i e_j^T - \frac{\Delta_1}{N-1}e_i(1_N^T - e_i^T), \text{ when } i \in \mathscr{F}, \text{ and } j \in \mathscr{N}; \tag{8a}$$

$$\mathbf{E}[L\otimes L]_{ij} = 0, \text{ when } i \in \mathscr{L}, \text{ and } j \in \mathscr{F}; \tag{8b}$$

$$\mathbf{E}[L\otimes L]_{ij} = \frac{-n^2 N}{(N-1)(l-1)}R_f - \frac{n^2 l}{(l-1)^2}R_l + \Delta_2 e_i e_j^T - \frac{\Delta_2}{l-1}e_i(1_l^T - e_i^T), \text{ when } i \in \mathscr{L}, \text{ and } j \in \mathscr{L}; \tag{8c}$$

where $\Delta_1 = \frac{n(N-n-1)}{(N-1)(N-2)}$ and $\Delta_2 = \frac{n(l-n-1)}{(l-1)(l-2)}$.

Substituting (7) and (8a) to (8c) in (4), we derive the matrix G in block form. We notice that the blocks have six cases: diagonal blocks G_{ii} can have $i \in \mathscr{F}$ or $i \in \mathscr{L}$ and off-diagonal blocks can have i and j belonging to either $\mathscr{F}$ or $\mathscr{L}$. The blocks of G when both i and $j \in \mathscr{F}$ are given as follows

$$\begin{aligned}\underset{(i\in\mathscr{F})}{G_{ii}} =& \theta_1 I_N + \theta_2 1_N 1_N^T + \theta_3 \hat{I}_f + \theta_4 1_f 1_N^T + \theta_5 1_N 1_f^T + \theta_6 \hat{I}_l + \theta_7 1_l 1_l^T + \theta_8 1_f e_i^T + \theta_9 1_N e_i^T + \theta_{10} e_i e_i^T + \\ & \theta_{11} e_i 1_f^T + \theta_{12} e_i 1_l^T + \theta_{13} 1_l e_i^T + \theta_{14} 1_N 1_l^T;\end{aligned} \tag{9a}$$

$$\begin{aligned}\underset{(i,j\in\mathscr{F},i\neq j)}{G_{ij}} =& \theta_{29} I_N + \theta_{30} 1_N 1_N^T + \theta_{31} \hat{I}_f + \theta_{32} 1_f 1_N^T + \theta_{33} 1_N 1_f^T + \theta_{34} \hat{I}_l + \theta_{35} 1_l 1_l^T + \theta_{36} e_i e_j^T + \theta_{37} 1_N e_j^T + \theta_{38} e_i 1_N^T \\ & + \theta_{39} e_i e_i^T + \theta_{40} 1_N e_i^T + \theta_{41} 1_f e_j^T + \theta_{42} e_j e_j^T + \theta_{43} e_j 1_f^T + \theta_{44} e_j 1_l^T + \theta_{45} 1_l e_j^T + \theta_{46} 1_N 1_l^T + \theta_{47} e_i 1_l^T;\end{aligned} \tag{9b}$$

where $\hat{I}_f, \hat{I}_l \in \mathbb{R}^{N\times N}$ are diagonal matrices and $\hat{I}_f$ has first f diagonal entries equal to one and the remaining zeros, and $\hat{I}_l$ has last l diagonal entries equal to one and the remaining zeros. For G_{ii}, when $i \in \mathscr{L}$, the form of (9a) is retained and the coefficients $\theta_1,\dots,\theta_{14}$ are replaced with $\theta_{15},\dots,\theta_{28}$. Similarly, for the off-diagonal blocks of G when $i\in\mathscr{F}$ and $j\in\mathscr{L}$; $i\in\mathscr{L}$ and $j\in\mathscr{F}$; and $i\in\mathscr{L}$ and $j\in\mathscr{L}$ the form remains the same as that of (9b) and the coefficients $\theta_{29},\dots,\theta_{47}$ become respectively $\theta_{48},\dots,\theta_{66}$; $\theta_{67},\dots,\theta_{85}$; and $\theta_{86},\dots,\theta_{104}$. The coefficients, θ_i for $i=1,\dots,104$ can be found in the Appendix.

It can be verified that G has at most twelve distinct eigenvalues and associated eigenspaces. The eigenvectors belonging to the eigenspaces have the form $v=[{v_1}^T{v_2}^T\cdots{v_N}^T]^T$ and satisfy the eigenvalue equation $Gv=\lambda v$. The twelve distinct eigenvalues are given as follows

$$\lambda_1 = \varepsilon^2\kappa_2 n + \sqrt{\varepsilon^4\kappa_3 n^2 - \varepsilon^3\kappa_5 n^2 + \varepsilon^2\kappa_4 n^2} - \varepsilon\kappa_1 n + 1, \tag{10a}$$

$$\lambda_2 = \varepsilon^2\kappa_7 n - \varepsilon\kappa_6 n + 1, \tag{10b}$$

$$\lambda_{3,12} = \gamma_1 + (r_{3,12})\gamma_5, \tag{10c}$$

$$\lambda_4 = \theta_1+\theta_6-\theta_{29}-\theta_{34}, \tag{10d}$$

$$\lambda_5 = \theta_{15}+\theta_{20}-\theta_{86}-\theta_{91}, \tag{10e}$$

$$\lambda_6 = \varepsilon^2\kappa_2 n - \sqrt{\varepsilon^4\kappa_3 n^2 - \varepsilon^3\kappa_5 n^2 + \varepsilon^2\kappa_4 n^2} - \varepsilon\kappa_1 n + 1, \tag{10f}$$

$$\lambda_{7,8} = \tau_9 + (r_{7,8})\tau_{10}, \tag{10g}$$

$$\lambda_9 = \theta_{15}+\theta_{17} - l\,(\theta_{48}+\theta_{50}) + (l-1)\,(\theta_{86}+\theta_{88}), \tag{10h}$$

$$\lambda_{10} = \theta_1+\theta_3-\theta_{29}-\theta_{31}, \tag{10i}$$

$$\lambda_{11} = 0. \tag{10j}$$

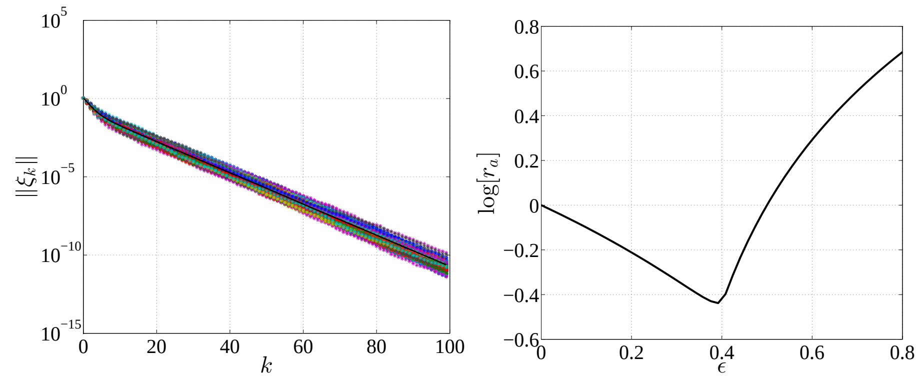
where the parameters $\kappa_1\dots\kappa_7, \gamma_1, \gamma_5, \tau_9, \tau_{10}$ are provided in the Appendix, $r_{3,12}$ are the two solutions of r of the quadratic equation: $(\gamma_2+\gamma_3+\gamma_7)+r(\gamma_4+\gamma_6+\gamma_8+\gamma_5-\gamma_1)-r^2(\gamma_5)=0$ and $r_{7,8}$ are the two solutions of r of the quadratic equation: $\tau_1+\tau_3+r(\tau_2+\tau_5+\tau_6+\tau_8-\tau_9)-r^2(\tau_{10})=0$. The explicit definitions of the eigenspaces associated to these eigenvalues are provided in the Appendix. Here we comment that r_1, r_6, ${r_2}^{(1)}$, and ${r_2}^{(2)}$ have highly intractable forms and hence we do provide the explicit expressions for these ratios. The eigenvectors corresponding to the eigenspaces $\Gamma^{(1)}$, $\Gamma^{(2)}$, and $\Gamma^{(6)}$ can be shown to satisfy the eigenvalue equation $Gv=\lambda v$ without the explicit expressions for these ratios. It can be verified that the eigenspaces $\Gamma^{(1)}$, $\Gamma^{(2)}$, and $\Gamma^{(6)}$ are mutually linearly independent, as are the triplet $\Gamma^{(7)}$, $\Gamma^{(8)}$, and $\Gamma^{(9)}$ and the pair $\Gamma^{(3)}$ and $\Gamma^{(12)}$. The remaining eigenspaces are mutually orthogonal to each other. Next, we find the eigenspace dimensions by counting the number of degrees of freedom for eigenvectors in each eigenspace, which are as follows: 1 for $\Gamma^{(1)}$, $\Gamma^{(2)}$, and $\Gamma^{(6)}$; $l-1$ for $\Gamma^{(3)}$; $2(l-1)(f-1)$ for $\Gamma^{(4)}$; $(l-1)(l-2)-1$ for $\Gamma^{(5)}$; $f-1$ for $\Gamma^{(7)}$, $\Gamma^{(8)}$, and $\Gamma^{(9)}$; $(f-1)(f-2)-1$ for $\Gamma^{(10)}$; $2N-1$ for $\Gamma^{(11)}$; and $2l-2$ for $\Gamma^{(12)}$. Since the eigenspaces are all pairwise linearly independent, their direct sum has dimension N^2. Hence, G has N^2 linearly independent eigenvectors and has a spectrum comprised of $\{\lambda_i\}_{i=1}^{12}$. The main result follows from Proposition 1.

Theorem 1. *For the NC leader-follower consensus protocol in* (1)*, with $f,l\geq 3$, $n\in\{1,\dots,\min\{f-1,l-1\}\}$, and the associated matrix G with eigenvalues in* (10a) *to* (10j)*, the asymptotic convergence factor r_a is given as*

$$r_a = \max_{i=1,\dots,12}\{|\lambda_i|\}. \tag{11}$$

4 Simulations and Discussion

Figure 1(a) presents Monte Carlo simulations for a network with $f=8$, $l=4$, $n=3$, $\varepsilon=0.1$ and fixed initial conditions. We observe as the disagreement system converges to zero, the magnitude of the disagreement vector decreases linearly on a logarithmic scale. We compute a best fit line in logarithmic scale over time



(a)

(b)

Fig. 1: (a) Magnitude of the disagreement vector for Monte Carlo simulations with constant initial conditions and $f = 8$, $l = 4$, $n = 3$, and $\varepsilon = 0.1$. Individual markers show two hundred different realizations and the black solid line depicts the average disagreement. (b) Base-ten logarithm of r_a varying with ε and $f = 8$, $l = 4$, $n = 3$.

steps $[10, 20]$ and find that the square of the disagreement norm decreases as $(0.789)^k$, thus confirming the analytical prediction in (11) which gives $r_a = 0.794$ for the same set of system parameters.

To study the dependence of the asymptotic convergence factor on model parameters, we choose a system with parameters $f = 8$, $l = 4$, and $n = 3$ and plot r_a varying with ε in Figure 1(b). We observe that the curve for r_a as we vary ε has a characteristic shape for all admissible values of N, f, l, and n. In particular, $\log[r_a]$ equals zero when $\varepsilon = 0$, decreases up to a certain negative value of ε, and then increases unbounded as $\varepsilon \to \infty$. By definition, convergence speed increases as r_a decreases and the systems with $r_a > 1$ do not converge to consensus. Therefore, we denote the maximum convergence speed as r_a^*, where r_a is minimum, and its corresponding persuasibility as ε^*.

To further explore the behavior of the asymptotic convergence factor on the system parameters, we consider three specific cases for a numerical study: (a) followers are twice the number of leaders $f = 2l$, (b) leaders are twice the number of followers $l = 2f$, and (c) number of leaders is same as that of the followers $l = f$. In Figure 2(a), Figure 2(b), and Figure 2(c), we fix N and n and plot the asymptotic convergence factor in base-ten logarithmic scale for the three aforementioned cases. In each of these plots, we observe that $\log[{r_a}^*]$ decreases as the proportion of leaders increases in each system, which means the maximum convergence speed increases as we increase the relative number of leaders with all other system parameters held constant. Moreover, increasing the proportion of leaders results in the decrease of ε^*, which indicates that in the presence of higher proportions of leaders, the agents must be less persuasible or in other words more stubborn to attain maximum convergence speed. However, increasing the relative number of leaders not always results in achieving faster convergence speed for system performance at a set persuasibility (or value of ε). Figure 2(a) demonstrates that increasing the proportion of leaders results in a slight decrease of convergence speed when $\varepsilon = 0.31$. Also, in Figure 2(b), and Figure 2(c), there exists ranges of ε ($[0.036, \infty)$ and $[0.293, \infty)$, respectively) where $\log[r_a]$ is not susceptible to different proportion of leaders.

Next, to investigate the effect of group size on these trends of the asymptotic convergence factor, we compare every case of Figure 2(a) with the corresponding case in Figure 2(b). We observe that when numerosity is increased proportionally with group size, both r_a^* and ε^* decrease. In other words, by increasing group size, we observe maximum convergence speed increases if numerosity is kept proportionally constant, but the agents must be more stubborn to achieve it. Further, in Figure 2(a) and Figure 2(c), we fix the nu-

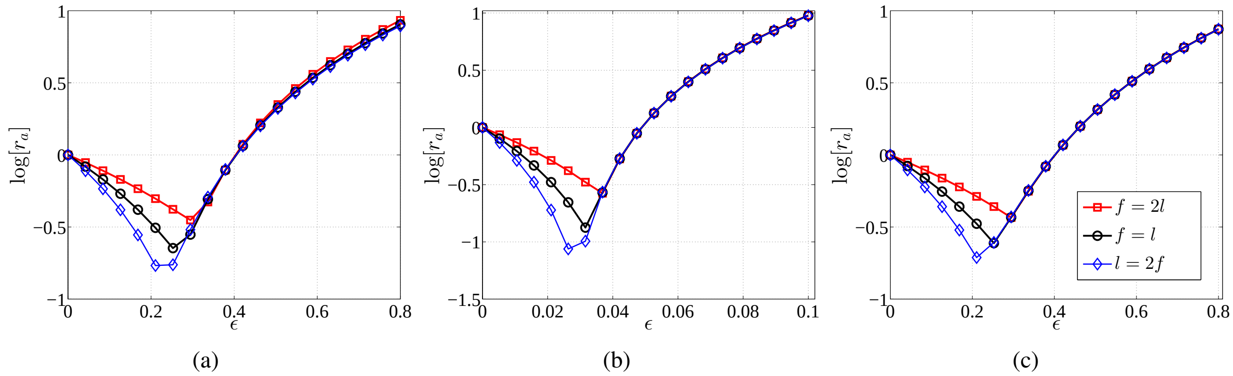


Fig. 2: Comparative study of the asymptotic convergence factor for three different proportions of leaders and followers and (a) $N = 18$ and $n = 4$, (b) $N = 180$ and $n = 40$, (c) $N = 180$ and $n = 4$.

merosity and increase the group size, which results in a slight decrease of maximum convergence speed and corresponding persuasibility. This is in support with the previous argument that as each agent interacts in a larger group, maximum convergence speed decreases if numerosity for each agent is not increased.

Finally, we study the dependence of asymptotic convergence factor on the numerosity of individuals keeping the other system parameters constant. Comparing each case of Figure 2(b) with that of Figure 2(c), we observe r_a^* and ε^* decrease between the figures at the constant group size. This demonstrates that, as we fix the group size and increase numerosity for each agent, the system achieves a faster maximum convergence speed at a significantly lower value of persuasibility. In other words, as a consequence of increasing numerosity, the agents are enabled with increasing information exchange among themselves, and this results in faster maximum convergence speed, while simultaneously requiring that the agents to be more stubborn.

5 Conclusion

In conclusion, we define a discrete-time leader-follower consensus protocol, where both leaders and followers negotiate their states over a stochastically-switching network. We determine a closed form expression of the asymptotic convergence factor, which measures the rate of convergence to consensus. Finally, we explore the dependence of the asymptotic convergence factor on model parameters, which are group size, proportion of leaders, numerosity, and persuasibility using numerical simulations. We find that the system achieves consensus faster with a higher proportion of representatives but this convergence rate is achieved only when all individuals are more stubborn during the decision making process. In addition, increasing the population size necessitates increasing the connectedness of each individual to maintain the system's ability to reach consensus, when the proportion of representatives is kept constant.

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6 APPENDIX

The eigenspaces of G in (9) are

$$\Gamma^{(1),(6)} = \{v \in \mathbb{R}^{N^2} : v_i = (\mu_1 - \mu_2)e_i + \mu_2 1_f + \mu_3 1_l : (\mu_1 - \mu_2) + f\mu_2 + l\mu_3 = 0, \frac{\mu_3}{\mu_2} = r_{1,6} \text{ when } i \in \mathscr{F}$$

$$\text{and } v_i = \mu_3 1_f + \mu_4 1_l : f\mu_3 + l\mu_4 = 0 \text{ when } i \in \mathscr{L}\}; \tag{12a}$$

$$\Gamma^{(2)} = \{v \in \mathbb{R}^{N^2} : v_i = (\mu_1 - \mu_2)e_i + \mu_2 1_f + \mu_3 1_l : (\mu_1 - \mu_2) + f\mu_2 + l\mu_3 = 0, \frac{\mu_2}{\mu_3} = r_2{}^{(1)} \text{ when } i \in \mathscr{F}$$

$$\text{and } v_i = \mu_3 1_f + \mu_4 1_l + (\mu_5 - \mu_4)e_i : f\mu_3 + l\mu_4 + (\mu_5 - \mu_4) = 0, \frac{\mu_3}{\mu_4} = r_2{}^{(2)} \text{ when } i \in \mathscr{L}\}; \tag{12b}$$

$$\Gamma^{(3)} = \{v \in \mathbb{R}^{N^2} : v_i = \sum_{k=f+1}^{N} \mu_k e_k : \sum_{j=f+1}^{N} \mu_j = 0 \text{ when} i \in \mathscr{F} \text{ and } v_i = \mu_i 1_f + \sum_{k=f+1}^{N} \beta_k^i e_k : \beta_{f+m}^{f+m} = (r_3)\,\mu_{f+m},$$

$$\beta_{f+m}^{f+n}(m \neq n) = -\left(\frac{f + r_3}{l-2}\right)\left(\mu_{f+n} + \mu_{f+m}\right) \text{ where } m,n \in \mathscr{L} \text{ when } i \in \mathscr{L}\}; \tag{12c}$$

$$\Gamma^{(4)} = \{v \in \mathbb{R}^{N^2} : v_i = \sum_{j=f+1}^{N} \mu_j^i e_j : \sum_{j=f+1}^{N} \mu_j^i = 0, \sum_{i=1}^{f} \mu_j^i = 0 \text{ when } i \in \mathscr{F} \text{ and } v_i = \sum_{j=1}^{f} \mu_j^i e_j : \sum_{j=1}^{f} \mu_j^i = 0,$$

$$\sum_{i=f+1}^{N} \mu_j^i = 0 \text{ when } i \in \mathscr{L}\}; \tag{12d}$$

$$\Gamma^{(5)} = \{v \in \mathbb{R}^{N^2} : v_i = 0 \text{ when } i \in \mathscr{F} \text{ and } v_i = \sum_{k=f+1}^{N} \mu_k^i e_k : \sum_{k=f+1}^{N} \mu_k^i = 0,\ \sum_{i=f+1}^{N} \mu_k^i = 0,\ \mu_i^i = 0 \text{ when } i \in \mathscr{L}\}; \tag{12e}$$

$$\Gamma^{(7),(8)} = \{v \in \mathbb{R}^{N^2} : v_i = (\mu_i)\,1_l + \sum_{k=1}^{f} \beta_k^i e_k : \beta_k^k = (r_{7,8})\,\mu_k,\ \beta_m^n(m \neq n) = -\left(\frac{l + r_{7,8}}{f-2}\right)(\mu_n + \mu_m)$$

$$\text{where } m,n \in \mathscr{F} \text{ and } \sum_{j=1}^{f} \mu_j = 0 \text{ when } i \in \mathscr{F} \text{and } v_i = \sum_{k=1}^{f} \mu_k e_k \text{ when } i \in \mathscr{L}\}; \tag{12f}$$

$$\Gamma^{(9)} = \{v \in \mathbb{R}^{N^2} : v_i = (\mu_i)\,1_l + \sum_{k=1}^{f} \beta_k^i e_k : \beta_k^k = 0, \beta_m^n(m \neq n) = -\left(\frac{l}{f}\right)(\mu_n - \mu_m) \text{ where } m,n \in \mathscr{F}$$

$$\text{and } \sum_{j=1}^{f} \mu_j = 0 \text{ when } i \in \mathscr{F} \text{ and } v_i = \sum_{k=1}^{f} \mu_k e_k \text{when } i \in \mathscr{L}\}; \tag{12g}$$

$$\Gamma^{(10)} = \{v \in \mathbb{R}^{N^2} : v_i = \sum_{k=1}^{f} \mu_k^i e_k : \mu_i^i = 0,\ \sum_{k=1}^{f} \mu_k^i = 0, \sum_{i=1}^{f} \mu_k^i = 0 \text{ when } i \in \mathscr{F} \text{ and } v_i = 0 \text{ when } i \in \mathscr{L}\}; \tag{12h}$$

$$\Gamma^{(11)} = \{v \in \mathbb{R}^{N^2} : v = \omega \otimes 1_N \text{ and } v = 1_N \otimes \omega, \omega \in \mathbb{R}^N\}; \tag{12i}$$

$$\Gamma^{(12)} = \{v \in \mathbb{R}^{N^2} : v_i = \sum_{k=f+1}^{N} \mu_k e_k : \sum_{j=f+1}^{N} \mu_j = 0 \text{ when } i \in \mathscr{F} \text{ and } v_i = \nu_i 1_f + \sum_{k=f+1}^{N} \beta_k^i e_k :$$

$$\beta_{f+n}^{f+m} = -\left(\frac{f}{l}\right)\left(\mu_{f+n} + \nu_{f+m}\right) \text{and } \sum_{j=f+1}^{N} \nu_j = 0 \text{ where } m,n \in \mathscr{L} \text{ when } i \in \mathscr{L}\}; \tag{12j}$$

The parameters used in (9) and in (10) are

$\alpha_1 = \varepsilon^2 n^2 \left(\frac{f-1}{(N-1)^2} + 1\right) - \varepsilon n;\ \alpha_2 = \alpha_1 + \frac{\varepsilon^2 n^2 N}{(N-1)^2};\ \alpha_3 = \frac{\varepsilon^2 l n^2}{(l-1)(N-1)} + \theta_6;\ \alpha_4 = \frac{\varepsilon^2 n^2 (f-2N)}{(N-1)^2} + \frac{\varepsilon n}{N-1};$
$\alpha_5 = \frac{\varepsilon^2 n^2 (f-N)}{(N-1)^2} + \frac{\varepsilon n}{N-1};\ \alpha_6 = \varepsilon^2 n^2 \left(\frac{f}{(N-1)^2} + \frac{l-2}{(l-1)(N-1)} - \frac{1}{l-1} - \frac{1}{N-1}\right) + \frac{\varepsilon n}{N-1};\ \alpha_7 =$

$$\frac{\varepsilon^2 l n^2\left(\frac{f}{N-1}-\frac{N-1}{l-1}+\frac{l-2}{l-1}-1\right)}{(l-1)N}+\frac{\varepsilon l n}{(l-1)N};\ \theta_1=\frac{(N-1)(1-\varepsilon n)}{N}-\frac{\varepsilon n(f-1)}{N(N-1)};\ \theta_2=\frac{\frac{\alpha_1 f}{N}+(f-1)\theta_{11}-\theta_1}{N};$$

$$\theta_3=\alpha_1+\theta_{11};\ \theta_4=-\left(\frac{\alpha_1}{N}+\theta_{11}\right);\ \theta_5=-\frac{\theta_3}{N};\ \theta_6=\frac{\varepsilon^2 l n^2\left(\frac{f-1}{N-1}+N-1\right)}{(l-1)N}-\frac{\varepsilon l n(N-1)}{(l-1)N};\ \theta_7=-\frac{\theta_6}{l};\ \theta_8=$$
$$-\frac{\Delta_1\varepsilon^2}{N};\ \theta_9=\frac{-f\theta_8-\theta_{10}}{N};\ \theta_{10}=\frac{\Delta_1\varepsilon^2}{N-1};\ \theta_{11}=-\frac{\Delta_1\varepsilon^2}{N(N-1)};\ \theta_{15}=\theta_1-\frac{\varepsilon n}{N-1};\ \theta_{16}=\frac{\alpha_2 f}{N^2}+\frac{f\theta_{11}}{N}-\frac{\theta_{15}}{N};$$
$$\theta_{17}=\alpha_2+\theta_{11};\ \theta_{18}=-\left(\frac{\alpha_2}{N}+\theta_{11}\right);\ \theta_{19}=-\frac{\theta_{17}}{N};\ \theta_{20}=\alpha_3+\theta_{26};\ \theta_{21}=-\left(\frac{\alpha_3}{l}+\theta_{26}\right);\ \theta_{23}=\frac{-f\theta_8}{N}-\frac{l\theta_{27}}{N}$$
$$-\frac{\theta_{24}}{N};\ \theta_{24}=\frac{\Delta_2\varepsilon^2 l}{(l-1)N};\ \theta_{26}=-\frac{\theta_{24}}{l};\ \theta_{27}=-\frac{\Delta_2\varepsilon^2}{N};\ \theta_{28}=\frac{(l-2)\theta_{26}}{N};\ \theta_{29}=\frac{2\varepsilon n}{N-1}-\frac{\frac{\varepsilon f n}{N-1}+1}{N};\ \theta_{30}=\frac{1}{N}\left(\frac{\alpha_4 f}{N}\right.$$

$$\left.+(f-1)\theta_{11}+\theta_{10}-\theta_{29}\right);\ \theta_{31}=\alpha_4+\theta_{11};\ \theta_{32}=-\frac{\alpha_4}{N}-\theta_{11};\ \theta_{33}=-\frac{\alpha_4+\theta_{11}}{N};\ \theta_{34}=\frac{\varepsilon^2 l n^2\left(\frac{f}{N}-2\right)}{(l-1)(N-1)}+\frac{\varepsilon l n}{(l-1)N};$$
$$\theta_{35}=-\frac{\theta_{34}}{l};\ \theta_{36}=\Delta_1\varepsilon^2;\ \theta_{37}=\frac{-f\theta_8-\theta_{10}-\theta_{36}}{N};\ \theta_{48}=\frac{1}{N}\left(\frac{\varepsilon n(N-f)}{(N-1)}-1\right);\ \theta_{49}=\frac{\frac{\alpha_5 f}{N}+f\theta_{11}+\theta_{10}-\theta_{48}}{N};$$
$$\theta_{50}=\alpha_5+\theta_{11};\ \theta_{51}=-\frac{\alpha_5}{N}-\theta_{11};\ \theta_{52}=-\frac{\alpha_5+\theta_{11}}{N};\ \theta_{53}=\theta_{26}+\theta_{72};\ \theta_{54}=-\left(\frac{\theta_{72}}{l}+\theta_{26}\right);\ \theta_{73}=-\frac{\theta_{72}}{l};$$
$$\theta_{56}=\frac{-f\theta_8-l\theta_{27}-\theta_{24}-\theta_{36}}{N};\ \theta_{68}=\frac{\frac{\alpha_5 f}{N}+(f-1)\theta_{11}-\theta_{48}}{N};\ \theta_{72}=\frac{\varepsilon l n}{(l-1)N}-\frac{\varepsilon^2 l n^2(N-f)}{(l-1)N(N-1)};\quad \theta_{96}=\frac{\Delta_2\varepsilon^2}{l-1};$$
$$\theta_{86}=\frac{\varepsilon n}{N}\left(-\frac{f}{(N-1)}+\frac{N-1}{(l-1)}+\frac{2-l}{(l-1)}+1\right)-\frac{1}{N};\ \theta_{87}=\frac{\frac{\alpha_6 f}{N}+f\theta_{11}-\theta_{86}}{N};\ \theta_{88}=\alpha_6+\theta_{11};\quad \theta_{91}=\alpha_7+\theta_{26};$$
$$\theta_{89}=-\left(\frac{\alpha_6}{N}+\theta_{11}\right);\ \theta_{90}=-\frac{\alpha_6+\theta_{11}}{N};\ \theta_{92}=-\left(\frac{\alpha_7}{l}+\theta_{26}\right);\ \theta_{93}=\Delta_2\varepsilon^2;\ \theta_{94}=\frac{-f\theta_8-l\theta_{27}-\theta_{24}-\theta_{93}}{N};$$
$$\theta_{103}=\frac{(l-2)\theta_{26}+\theta_{96}}{N};\ \theta_{12}=\theta_{40}=\theta_{43}=\theta_{44}=\theta_{59}=\theta_{81}=\theta_{82}=\theta_{11};\ \theta_{13}=\theta_{14}=\theta_{25}=\theta_{45}=\theta_{46}=\theta_{47}=\theta_{62}$$
$$=\theta_{66}=\theta_{74}=\theta_{76}=\theta_{77}=\theta_{78}=\theta_{83}=\theta_{84}=\theta_{85}=\theta_{95}=\theta_{100}=0;\ \theta_{22}=\theta_{41}=\theta_{60}=\theta_{79}=\theta_{98}=\theta_8;$$
$$\theta_{39}=\theta_{42}=\theta_{58}=\theta_{80}=-\theta_{38}=-\theta_{57}=\theta_{10};\ \theta_{55}=\theta_{36};\ \theta_{61}=\theta_{99}=\theta_{24};\ \theta_{63}=\theta_{97}=\theta_{101}=\theta_{26};$$
$$\theta_{65}=\theta_{28};\ \theta_{67}=\theta_{48};\ \theta_{69}=\theta_{50};\ \theta_{70}=\theta_{51};\ \theta_{71}=\theta_{52};\ \theta_{75}=\theta_9;\ \theta_{102}=\theta_{64}=\theta_{27};\ \theta_{104}=-\theta_{96};$$
$$K_1=2n(N-1)(-l+N-1)\left(-l^2N-l(N+2)+N^3-3N^2+3N-2\right)+n^2\left(l^4(N-2)-2l^3N\right.$$
$$\left.-l^2\left(2N^3-6N^2+5N+2\right)+2l\left(N^3-N^2+N-2\right)+(N-2)\left(N^2-N+1\right)^2\right)+(N-2)(N-1)^2(l-N+1)^2;$$
$$K_2=n\left((l-N)\left(l^2(N-2)-lN-N^3+3N^2-3N+2\right)-2l\right)+(N-1)N(-l+N-2)(-l+N-1);$$
$$K_3=\left(N^2-3N+2\right)\left(l^2n+l(n-N+1)+n\left(N^2-N+1\right)+(N-1)^2\right);\ \kappa_1=\frac{f^2+f(3l-2)+2l(l-2)}{(N-2)(N-1)};$$
$$\kappa_2=\frac{K_3}{2(N-2)(N-1)^3};\ \kappa_3=\frac{K_1}{4(N-2)(N-1)^4};\ \kappa_4=\frac{f^2}{(N-1)^2};\ \kappa_5=\frac{K_2}{(N-2)(N-1)^3};\ \kappa_6=\frac{2l}{l-1};$$
$$\kappa_7=\frac{ln}{(l-1)^2}+n+1;\ \kappa_8=\kappa_1-\kappa_6;\ \kappa_9=\kappa_7-\kappa_2;\ \kappa_{10}=\kappa_9^2-\kappa_3;\ \kappa_{11}=\kappa_8^2-\kappa_4;\ \kappa_{12}=\kappa_5+2\kappa_8\kappa_9;$$
$$\gamma_1=f\left(\theta_{103}+\theta_{19}-\theta_{26}-\theta_{28}-\theta_{90}\right)+\theta_{15}+\theta_{17}-\theta_{86}-\theta_{88};\ \gamma_2=f\left(\theta_{103}+\theta_{19}-\theta_{21}-\theta_{26}-\theta_{28}-\theta_{90}+\theta_{92}\right);$$
$$\gamma_3=f\left(-\theta_{26}+\theta_{48}+\theta_{72}-\theta_{86}-\theta_{91}\right);\ \gamma_4=\theta_{15}+\theta_{20}-\theta_{86}-\theta_{91};\ \gamma_5=\gamma_6=\theta_{23}-\theta_{26}-\theta_{94};\ \gamma_7=-2f\theta_{96};$$
$$\gamma_8=-\theta_{93}-\theta_{96};\ \tau_1=l\left(-\theta_{29}-\theta_{31}+\theta_{48}+\theta_{50}\right);\ \tau_2=\theta_1-\theta_{29}-\theta_{31}+\theta_3;\ \tau_3=\theta_{33}l-\theta_5 l;\ \tau_5=-\theta_{10}-\theta_{36};$$
$$\tau_6=-\theta_{11}-\theta_{37}+\theta_9;\ \tau_8=\theta_{10};\ \tau_9=\theta_{15}+\theta_{17}-l(\theta_{48}+\theta_{50})+(l-1)(\theta_{86}+\theta_{88});\ \tau_{10}=\theta_{10};$$