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Nonlinear Analysis of Pull-In Voltage of Two Micro-Cantilever Beams

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Abstract– Silicon-based micro-cantilever beams, due to their simplicity of fabrication and versatility are candidates for a large range of engineering applications. Their static and dynamic behaviour under the influence of various types of loadings were investigated extensively during last two decades. The object of the present work is to examine the non-linear differential equations that models the dynamic performance of single and double cantilever beams subjected to an electrostatic field. The main focus of the study is the evaluation of the critical pull-in voltage, i.e. the voltage closed to the snap-on of the micro-structure. Pull-in voltage, a topic of great concern in MEMS literature due to the wide range of application where cantilever-type structures interact with electric field, is investigated from a theoretical point of view. The highly stiff one degree of freedom non-linear differential equation modelling the dynamic behaviour of the cantilever under electrostatic loading can be satisfactorily studied by adopting a stiffness model for the considered designs. Only the ISODE Maple algorithm can satisfactory solve numerically the fore mentioned equation due to its adaptive time-step selection mechanisms. The stiffness model is chosen from the ones found in literature [insert references here]. However, the most suitable stiffness model for the static study proves to be different from the optimum model involved in the dynamic study. The influence of excitation voltage on pulling voltage and the effect of structural damping on large deflection are investigated numerically. A closed-form time response to step-voltage is derived and pull-in voltage calculated for an undamped system and compared to the one analytically determined by solving the reduced form of the non-linear modelling ODE.

Keywords**:** pull-in voltage, micro-cantilever, Lie symmetries,

1. Introduction

Pull-in voltage is an important parameter for silicon-type micro-cantilever beams subjected to electric fields. Its value gives important information regarding the limitation of the investigated system, that is, about the moment when the structure becomes unstable. Empirically it is observed (Schiele et al., 1998a), that the beam, attracted by the fixed electrode, approaches the pull-in voltage when it reaches a position that corresponds to 2/3 of the original gap between the beam and the fixed electrode. There are many articles studying the operation of micro-fabricated cantilever structures interacting with different types of electric fields, both from a theoretical and experimental point of view.

In general, the pull-in voltage is found from the differential equation that describes the dynamic behaviour of the cantilever beam in an electrostatic field. The governing differential equation can be obtained using either Hamiltonian (Hu, et al., 2004) or energy (Chen, et al., 2009) methods. Due to the high non-linearity, several approaches are employed with the purpose of the simplification of these equations. In literature, finite element method is one of the most extensive tactics adopted for the numerical determination of the pull-in voltage (Busta, et al., 2001). Taylor series are used to linearize the governing differential equation (Younis, et al., 2003). Perturbation method (Zhang, et al., 2002) and Runge-Kutta algorithms are other modalities of solving the derived Duffing equations that model the dynamic behaviour of the cantilever beam subjected to both electric field and harmonic excitation. Another approach assumes small deflections of the structures involved and the results are compared with the experimental data (Schiele, et al., 1998b). Dimensionless continuum beam theory or orthogonal functions type Taylor series (Kuang, et al., 2004) can be employed in the derivation of the governing equations describing the dynamic performance of the micro-cantilever. In the small deflection approach, the error is found to be directly proportional with the length of the beam and decreases when the structure is excited by a potential significantly below the snap-on value (Hung, et al., 1999). The effects of the width and thickness of the cantilever on its resonant frequency are also the subject of several theoretical and experimental studies (Chowdhury, et al., 2005).

2. Theory

The dynamics of an electrostatic cantilever-type actuator, using the lump mass hypothesis is described by the equation (1). Figure 1 illustrates the considered simplified model by using a mass-spring-damper system:

$$\frac{d^2y(t)}{dt^2} + 2\xi\omega_n\frac{dy(t)}{dt} + \omega_n^2 y(t) = \frac{f(t)}{m} \tag{1}$$

In equation (1) f(t) represents the electrostatic force, m the mass of the beam, y(t) the deflection and ξ is the damping factor $\left(\frac{C}{m} = 2\xi\omega_n\right)$.

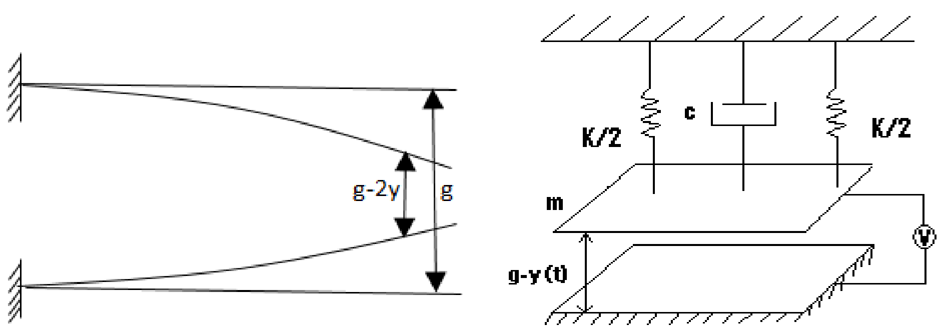

Fig. 1. The schematic of a mass-spring damper system of a beam

The value of the force exerted due to the electrostatic effect between the two parallel surfaces is significant in micro-structures. Its expression can be obtained from the energy balance and is given by:

$$f(t) = \frac{\varepsilon_0 A V^2}{2(g-2y(t))^2} \tag{2}$$

In equation (2) ε_0 represents the absolute permittivity of the medium between the surfaces, g is initial distance between beam and substrate, A is area of the beam and V is the voltage.

The equation (1) can be rewritten as:

$$\frac{d^2y(t)}{dt^2} + 2\xi\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \frac{\varepsilon_0 A V^2}{2m\left(g-2y(t)\right)^2} \tag{3}$$

Where the initial conditions (initial speed at reference position and time) for this ODE are assumed to be the following:

$$y|_{t=0} = 0 \qquad \text{and} \qquad \left.\frac{dy}{dt}\right|_{t=0} = v_0 \tag{4}$$

There is no analytical formulation in our knowledge to express the solution of the non-linear equation (4) in closed form. The micro dimensions of the structures involved contribute to the stiffness of the ODE. The current approach used to solve the equation (3) is the numerical one. The present study proposes a method of reduction of the order of the governing equation using Lie symmetry method, transforming the second order ODE into a first order one that can subsequently be solved in an easier manner. In the subsequent paragraphs, the terminology involved by the use of Lie symmetry method requires is presented.

The point symmetric transformations require that each point (x, y) on a specific curve moves into a point (x_1, y_1).

$$x_1 = \phi(x, y, \alpha) \qquad y_1 = \psi(x, y, \alpha) \tag{5}$$

Where ϕ, ψ are diffeomorphism (C^∞). A transformation that preserves the shape of a given curve and it maps this curve on itself, is called symmetry. The transformation (5) that satisfies the group properties is called a one-parameter group while α is called the parameter of the group.

For a one-parameter group an infinitesimal transformation is defined as :

$$Uf = \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y} \tag{6}$$

Where:

$$\eta(x, y) = \left.\frac{\partial \psi}{\partial \alpha}\right|_{\alpha=0} \qquad \xi(x, y) = \left.\frac{\partial \phi}{\partial \alpha}\right|_{\alpha=0} \qquad f = f(x, y) \tag{7}$$

The transformation operator on the function is defined as U. The necessary and sufficient condition for a group to be a symmetry transformation for a function $f = f(x, y)$ is:

$$Uf = 0 \tag{8}$$

The condition (8) is further used to calculate the infinitesimal transformations for the ODE (3). The most used procedure in calculating the Lie symmetry is the prolonged vector method [ ].

Let us consider a second-order ODE given by:

$$\frac{d^2y}{dx^2} = \omega(x, y, \frac{dy}{dx}) \tag{9}$$

If an infinitesimal group is applied as an operator on the equation (9), both functions ξ and η defined in (7), must satisfy the following equality [ ]:

$$\eta_{xx} + \left(2\eta_{xy} - \xi_{xx}\right)\acute{y} + \left(\eta_{yy} - 2\xi_{xy}\right)\acute{y}^2 - \xi_{yy}\acute{y}^3 + \left(\eta_y - 2\xi_x - 3\xi_y\acute{y}\right)\omega = \tag{10}$$
$$= \omega_x + \eta\omega_y + \left((\eta_x - \xi_x)\acute{y} - \xi_y\acute{y}^2\right)\omega_{\acute{y}}$$

Subsequently, the equality (10) can be decomposed into a system of partial differential equations, and ξ and η can be calculated. For our case, taking in consideration that most significant Lie symmetries including rotation, translation and scaling could be found from the following equations:

$$\xi = C_1 + C_2 x + C_3 y \tag{11}$$
$$\eta = C_4 + C_5 x + C_6 y$$

By substituting the equations (11) in (10) gives:

$$\alpha C_6\acute{y} - 2C_2\alpha\acute{y} - 3C_3\alpha\acute{y}^2 + \beta C_6 - 2\beta C_2 y - 3\beta C_3 y\acute{y} = \beta C_4 + \beta C_5 x + \beta C_6 y +$$
$$+\alpha C_2 + \alpha C_6 - C_2\alpha\acute{y} - C_3\alpha\acute{y}^2 \tag{12}$$

Where, $\alpha = -2\xi\omega_n$, $\beta = -\omega_n^2$

The coefficients of $\acute{y}^2$ in left hand and right hand terms must be equal:

$$-3C_3\alpha = -C_3\alpha \tag{13}$$

Therefore:

$$C_3 = 0 \tag{14}$$

Following the same procedure, the coefficients of y' in left hand and right hand terms are equated:

$$\alpha C_6 - 2\alpha C_2 - 3\beta C_2 = \alpha C_6 - \alpha C_2 \tag{15}$$

By simplification:

$$-\alpha C_2 - 3\beta C_2 = 0 \tag{16}$$

$$C_2 = 0 \tag{17}$$

The coefficients of x in left hand and right hand terms must also be equal, then:

$$C_5 = 0 \tag{18}$$

Considering the expressions (14), (17) and (18), the equation (12) becomes:

$$\beta C_6 y = \beta C_4 + \beta C_6 y \tag{19}$$

With these considerations:

$$C_4 = 0 \tag{20}$$

From the above calculations, it can be concluded that the equation (3) has the following infinitesimal:

$$Uf = f_x \tag{21}$$

Any pair of functions $r(x,y), s(x,y)$ satisfying the following conditions forms canonical coordinates:

$$\begin{gathered} \xi(x,y)r_x + \eta(x,y)r_y = 0 \\ \xi(x,y)s_x + \eta(x,y)s_y = 1 \\ \begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix} \neq 0 \end{gathered} \tag{22}$$

The equation (21) satisfies the conditions (22) and therefore, $\xi(x,y) = 1, \eta(x,y) = 0$.

The canonical coordinates for a function $f(x,y)$ can be found from the characteristic equation (Changizi, 2011):

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)} = ds \tag{23}$$

The solution of the above ODE is $r(x,y)$:

$$\frac{dy}{dx} = \frac{\eta(x,y)}{\xi(x,y)} \tag{24}$$

and $s(x,y)$ will be:

$$S(r,x) = \left(\int \frac{dx}{\xi(x,y(r,x))}\right)\Big|_{r=r(x,y)} \tag{25}$$

Through adequate selection of variables, the order of the ODE can be reduced. From (23), (24) and (25) the canonical coordinates can be calculated as:

$$(r,s) = (y,t) \tag{26}$$

Considering:

$$r(y,t) = y \tag{27}$$

The function ν is defined as:

$$\nu = \frac{1}{\frac{dy}{dt}} \tag{28}$$

The equation can be expressed by contact form (Changizi, 2011) as:

$$\frac{d\nu}{dr} = -\frac{\frac{d^2y}{dt^2}}{(\frac{dy}{dt})^2} \tag{29}$$

or:

$$\frac{d^2y}{dt^2} = -\nu^{-2}\frac{d\nu}{dr} \tag{30}$$

Using the relations (23), (24), (25) the canonical coordinates and further $\nu = \frac{ds}{dr}$ can be calculated. By considering ν as a new variable and substituting in the ODE, the new generated ODE will have one order less than the original one.

Substituting (28), (29) and (30) in (3) yields:

$$-\frac{\mathrm{dv}}{\mathrm{dr}} + 2\xi\omega_{\mathrm{n}}\mathrm{v}^2 + \left(\omega_{\mathrm{n}}^2\mathrm{r} - \frac{\varepsilon_0 \mathrm{AV}^2}{2\mathrm{m}(\mathrm{g}-2\mathrm{r})^2}\right)\mathrm{v}^3 = 0 \tag{31}$$

The equation (31) is a first order ODE with $\nu(0) = \frac{1}{\mathrm{v}_0}$ as an initial condition.

According the recent investigations there is no one-parameter group that satisfies the symmetric condition (31). For this reason, no analytical solution for this ODE can be formulated. One can show that there is no transformation of scaling or rotation symmetry for (31) (Changizi, 2011). This equation has a singularity (where $r = g$) and the integration in closed form becomes impossible. Therefore, the numerical method approach is used in solving the differential equation.

2. Results

For both scenarios (one beam and two-beam setups) a numerical analysis is performed. The constructive parameters for the polysilicon beams are 200μm length, 20μm with and 2μm thickness with a Young modulus of 169 MPa and gap distance is 10μm. For a more detailed insight in the method of analysis one can refer to (Changizi, 2011).

The value of pull-in voltage is calculated by assuming that nonlinear part of the equation (31) to be zero, as follows:

$$\mathrm{V} = \sqrt{\frac{2\mathrm{my}}{\varepsilon_0 \mathrm{A}}}(\mathrm{d} - 2\mathrm{y})\omega_{\mathrm{n}} \tag{32}$$

Pulling voltage determined numerically from the bellow graphs is 129.055 V for a two-beam setup and 182.511 V for one beam. The pull-in value calculated from (32) shows that, for the two-beam setup, the error of the exact solution with respect to the numerical solution is 0.104%. For the single beam, using almost similar equation the calculated error increases slightly to 0.449 %. The value of the deflection used in the equation (32), was calculated through interpolation from figure 1.

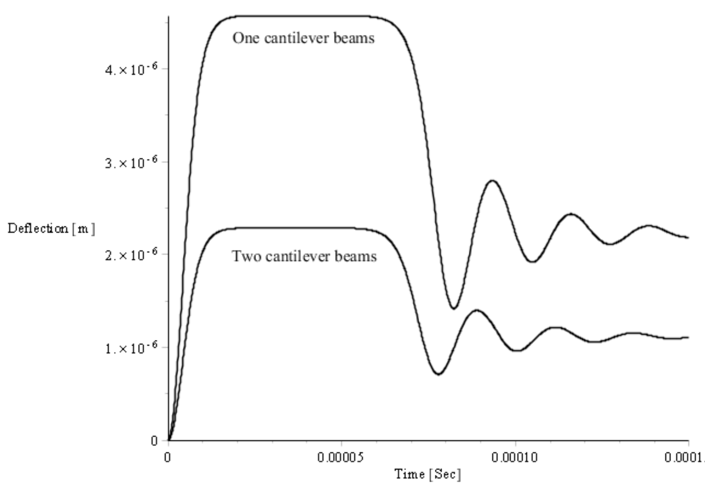

Fig. 2. The variation of the deflection for one beam and two-beam scenarios.

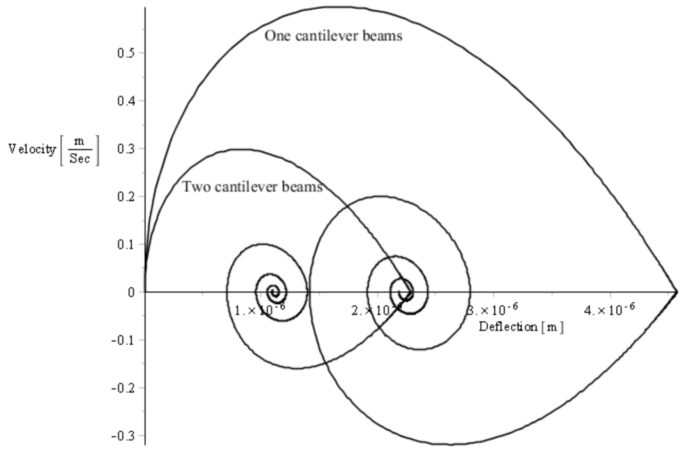

Fig. 3. Phase diagram of beams

3. Conclusions

The dynamic behaviour of a micro-cantilever beam under the influence of electric field, excited by an electric potential close to the pull-in voltage was investigated analytically and the results validated experimentally with data from the literature. A particular exact solution of the governing equation was found using the Lie symmetry method, by reducing the order of the initial ODE.

The pull-in voltage of one and two-beam setups was determined both analytically and numerically and the error between the two methods calculated.

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