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General Synchronization of Cascaded Boolean Networks Within Different Domains of Attraction

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Abstract- In this paper, we consider the synchronization phenomenon and related cyclic behaviours in Cascade Boolean networks (CBNs) within different domains of attraction (DAs). The results reveal that CBNs can exhibit diverse cyclic relations such as partial synchronization, completely synchronization, anti-synchronization, intermittent synchronization on different combinations of their DAs and initial states.

Keywords: Cascade Boolean networks, Cyclic pattern, General synchronization, Semi-tensor product

1 Introduction

In systems biology, Boolean networks (BNs) is an important model to describe the operation of gene regulatory networks (Kauffman 1969). A BN consists of a directed graph with nodes that represent genes or other elements. Each node assumes only two states: ”on” or ”off”, referring occurrence of a gene transcription or not. In a BN, every node gets input from its neighbouring nodes and updates its state simultaneously according to their interaction described by Boolean functions. Despite its simplicity, BN has found many applications in biological and engineering problems, such as (Stillman et al. 1996), (Romond et al. 1999) and (Heidel et al. 2003).

An interesting issue in study of BNs addresses the synchronization of coupled BNs due to its potential applications. For instance, in the cortical networks in a brain, synchronization among related parts is required to make the entire system capable of performing certain functions (Zhou et al. 2007), (Garcia-Ojalvo et al. 2004). Existing works on synchronization of BNs, e.g., (Morelli et al. 1998), (Morelli et al. 2001), (Ho et al. 2001), (Hung et al. 2006), and (Hung 2011), mostly consider random systems and are based on numerical simulation. Recently, Hong and Xu in (Hong et al. 2010), (Xu et al. 2013), and Li in (Li et al. 2012a) presented analytical results for synchronization of deterministic BNs, using the recently developed theory of semi-tensor product (STP) of matrices by Cheng and Qi in (Cheng et al. 2010, 2011a). This approach allows for converting a logical function into its equivalent algebraic form and hence facilitating analysis of BNs by means of the conventional control system theory. In this setting, many results have been obtained for the synchronization of BNs connected in ”drive-response” configuration, (Li et al. 2012b), (Li et al. 2012c), (Li et al. 2013).

In this paper, we consider general synchronization of CBNs within specified DAs. Our discussion is based on algebraic representations of BNs and gives rigorous analysis of the system dynamics. We give synchronization criterion and show that CBNs can exhibit richful cyclic behaviours depending on different combination of DAs of the system.

This paper is organized as follows. Section 2 gives brief review of theory of STP and Section 3 presents the CBNs model. In Section 4, we establish a synchronization criterion, and In Section 5 we discuss the cyclic behaviours of CBNs. A brief conclusion is drawn in Section 6.

2 Preliminaries

In this section, we briefly introduce the STP theory and useful properties (Cheng et al. 2010).

Let A an $n \times m$ matrix and B a $p \times q$ matrix, the semi-product of A and B is defined as

$$A \ltimes B = (A \otimes I_{l/m})(B \otimes I_{l/p}),$$

where $\otimes$ is the Kronecker product and $l = lcm(m,p)$ is the least common multiple of m and p. Clearly, it is reduced to the conventional matrix product when $m = p$. We have the following properties of STP:

1. If x, y are column vectors, then $x \ltimes y = x \otimes y$.

2. If A is a matrix, x is an n dimensional column vector, then $x \ltimes A = (I_n \otimes A) \ltimes x$.

3. If x is an n dimensional column vector, y is an m dimensional column vector, then $y \ltimes x = W_{n,m} \ltimes x \ltimes y$.

4. If $x \in \Delta_n$, then $x \ltimes x = \Phi_n x$.

We can have a matrix expression for a logic variable by using STP technique. To see this, we assign logical value with a vector by letting $T = 1 \sim \delta_2^1$ and $F = 0 \sim \delta_2^2$, where $\delta_2^1 = [1 \ \ 0]^T$ and $\delta_2^2 = [0 \ \ 1]^T$. Then a logical variable $A(t)$ can assume vector values as below:

$$A(t) \in D := \Delta_2 = \{\delta_2^1, \delta_2^2\}.$$

Any logical function $L(A_1,\ldots,A_n)$ with logical arguments $A_1,\ldots,A_n$ can be expressed in a multi-linear form as

$$L(A_1,\ldots,A_n) = M_L A_1 A_2 \cdots A_n,$$

where $M_L \in L_{2\times 2^n}$ is uniquely determined by $L(A_1,\ldots,A_n)$ and referred to as the structure matrix of L, see (Cheng et al. 2010).

3 Model

The CBNs we considered can be expressed as

$$\begin{cases} u_1(t+1) & = g_1(u_1(t),\ldots,u_n(t)) \\ \quad\vdots & \qquad\qquad \vdots \\ u_n(t+1) & = g_n(u_1(t),\ldots,u_n(t)) \end{cases} \tag{1}$$

$$\begin{cases} x_1(t+1) & = f_1(x_1(t),\ldots,x_n(t),u_1(t),\ldots,u_m(t)) \\ \quad\vdots & \qquad\qquad \vdots \\ x_n(t+1) & = f_n(x_1(t),\ldots,x_n(t),u_1(t),\ldots,u_m(t)) \end{cases} \tag{2}$$

where, $u_j \in D, j = 1,2,\ldots,m$ and $x_i \in D, i = 1,2,\ldots,n$ represent node state variables of BNs (1) and (2), respectively; g_j, f_i are Boolean functions, $g_j : D^m \longrightarrow D, j = 1,2,\ldots,m$; $f_i : D^{n+m} \longrightarrow D, i = 1,2,\ldots,n$; $t \in N = \{1,2,\ldots\}$.

By invoking the STP technique, we can convert (1) and (2) into the following algebraic form

$$\begin{cases} u(t+1) = & Gu(t), u \in D^m \\ x(t+1) = & Lu(t)x(t), x \in D^n \end{cases} \tag{3}$$

where,

$$u(t) = u_1(t)\cdots u_m(t) \in \Delta_{2^m},$$
$$x(t) = x_1(t)\cdots x_n(t) \in \Delta_{2^n}.$$

$G \in \mathscr{L}_{2^m \times 2^m}$, $L \in \mathscr{L}_{2^m \times 2^{n+m}}$. $Lu(t)$ is the input-determined transition matrix (Cheng et al. 2010a). For simplicity, we assume that the BN (1) has the same number of nodes as BN (2) does, i.e., $m = n$. Let $u(t,u_0)$ be the solution of (1) with the initial value $u_0 \in (\Delta_2)^n$, and $x(t,x_0,u_0)$ the corresponding solution of (2) with the initial value $x_0 \in (\Delta_2)^n$. Notice that the solution of (2) is determined by both the initial values x_0 and u_0. Therefore, different combinations of the DA of BNs (1) and (2) in which the initial states are chosen may give rise different dynamic behaviour in the coupled systems, thus resulting in diverse dynamic behaviours. The primary objective of this paper is to analyse and classify such kind of diverse dynamics patterns with respect to the DAs of the coupled systems.

Assume that BN (1) has p distinct DAs, $S_1,\ldots,S_p$, $p \in N$ and for a fixed DA S_i, $1 \le i \le p$, BN (2) has q_i corresponding input-determined DAs, $S_{i1},\ldots,S_{iq_i}$. Because of the finite state spaces of BNs, such DAs exist and can be calculated readily (Cheng et al. 2010).

Let S_i be a DA of BN (1), S_{ij} be a related input-determined DA of BN (2), $1 \le j \le q_i$. If there is a positive integer k, such that for any $u_0 \in S_i$ and $x_0 \in S_{ij}$,

$$x(t,x_0,u_0) = u(t,u_0), t \ge k,$$

then we call that the synchronisation of BNs (1) and (2) with respect to the given DAs occurs.

4 Main result

To establish our main result, we make use of block-diagonal form of BNs. By (Fornasini et al. 2013), there is a state transformation

$$\tilde{u}(t) = Pu(t),$$

such that the system (1) can be changed into the form

$$\tilde{u}(t+1) = \tilde{G}\tilde{u}(t).$$

where the new state transition matrix $\tilde{G}$ is a block-diagonal matrix,

$$\begin{aligned}\tilde{G} &= PGP^{-1}\\ &= blockdiag[D_1, D_2, \ldots, D_p]\\ &= \begin{bmatrix} D_1 & & & \\ & D_1 & & \\ & & \ddots & \\ & & & D_p \end{bmatrix},\end{aligned}$$

with

$$D_i = \begin{bmatrix} N_i & 0 \\ T_i & C_i \end{bmatrix} \in \mathscr{L}_{n_i \times n_i}, i = 1,2,\ldots,p. \tag{4}$$

The matrix P corresponds to a so-called change of basis in the vector space of the logic functions of $u_1,\ldots,u_n$ (Cheng et al. 2011a).

From (Fornasini et al. 2013), the Block matrices $D_1,\ldots,D_p$ describes p DAs, $S_1,\ldots,S_p$ of BN (1) along with the dynamical properties, including attractors or cycles. $C_i \in \mathscr{L}_{k_i \times k_i}$ is the permutation matrix of i-th domain and its dimension represents the cycle length k_i. $N_i \in \mathscr{L}_{(n_i-k_i)\times(n_i-k_i)}$ is a nilpotent matrix. $T_p := max_{i\in[1,r]}(n_i-k_i)$ represents the maximum time length of transition process of BN (1), namely, after T_p steps, every trajectory will enter some limit cycle C_i. For details, please refer to (Fornasini et al. 2013).

Further, we set

$$y(t) = y_1(t)\cdots y_n(t)y_{n+1}(t)\cdots y_{2n}(t),$$

where

$$y_i(t) = u_i(t), i = 1,2,\ldots,n,$$

$$y_{n+j}(t) = x_j(t), j = 1,2,\ldots,n.$$

Then multiplying both sides of equation (3), we have

$$u(t+1)x(t+1) = Gu(t)Lu(t)x(t)$$

i.e.,

$$\begin{aligned} y(t+1) &= G(I_{2^m}\otimes L)u^2(t)x(t) \\ &= G(I_{2^m}\otimes L)\Phi_m u(t)x(t) \\ &= G(I_{2^m}\otimes L)\Phi_m y(t). \end{aligned}$$

Let

$$L_1 = G(I_{2^m}\otimes L)\Phi_m,$$

we get

$$y(t+1) = L_1 y(t).$$

Similarly, we can change BN (3) into a block-diagonal form by using a state transformation

$$\tilde{y}(t) = P_1 y(t),$$

and get

$$\tilde{y}(t+1) = \tilde{L_1}\tilde{y}(t).$$

where,

$$\begin{aligned} \tilde{L_1} &= P_1 L_1 P_1^{-1} \\ &= blockdiag[D_1, D_2, \ldots, D_{p'}] \\ &= \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_{p'} \end{bmatrix}, \end{aligned}$$

with

$$D_i = \begin{bmatrix} N_i & 0 \\ T_i & C_i \end{bmatrix} \in \mathscr{L}_{n_i\times n_i}, i = 1,2,\ldots,p'. \tag{5}$$

Similarly, the block matrices $D_1,\ldots,D_{p'}$ describe p' DAs of BN (3) along with the dynamical properties including attractors and cycles. $C_i \in \mathscr{L}_{k_i\times k_i}$ is the permutation matrix of i_{th} domain. Its dimension represents the cycle length; $N_i \in \mathscr{L}_{(n_i-k_i)\times(n_i-k_i)}$ is a nilpotent matrix. $T_{p'} := max_{i\in[1,l_1]}(n_i-k_i)$ represents the maximum time length of transition process of BN (3), namely, after $T_{p'}$ steps, every trajectory of BN (3) will enter some limit cycle C_i.

For unidirectional connectivity of BN (1) and BN (3) (Cheng et al. 2009), we have the following correspondence between block matrices of BNs (1) and (3).

$$D_i \longrightarrow \{D\}_i,$$

where, $\{D\}_i = \{D_{i1}, D_{i2}, \ldots, D_{iq_i}\}_i$ is obtained by dividing $D_1, \ldots, D_{p'}$ of $\tilde{L_1}$ into i non-overlapping groups, D_i and $\{D\}_i$ are of one-to-one correspondence. Assume the number of blocks to be $\{D\}_i$ as q_i. We give the following algorithm to determine the correspondence between D_i and $\{D\}_i$.

For any $D_i \in \tilde{L_1}$, $i = 1, 2, \ldots, p'$, we can take any column vector δ_i in the position of D_i and revert it to its original coordinate

$$\delta_{i0} = P_1^{-1} \delta_i.$$

Obviously, δ_{i0} can be separated as a STP product of two vectors of δ_{u0} and δ_{x0}, which represents the state of BNs (1) and (2), respectively.

$$\delta_{i0} = \delta_{u0} \ltimes \delta_{x0}.$$

By changing δ_{u0} to δ_u under new coordinate system, we have

$$\delta_u = P\delta_{u0}.$$

If the column vector of $\tilde{G} \ltimes \delta_u$ is located in the position of D_i, then the related block of D_i is D_i.

Note that by reverting C_i and C_{i_j} to their original coordinates, we have

$$P^{-1} \begin{bmatrix} 0 \\ C_1 \\ 0 \end{bmatrix} = \left[\delta_{cu_1} \delta_{cu_2} \cdots \delta_{cu_k} \right],$$

$$P_1^{-1} \begin{bmatrix} 0 \\ \tilde{C}_1 \\ 0 \end{bmatrix} = \left[\delta_{cy_1} \delta_{cy_2} \cdots \delta_{cy_{k_1}} \right].$$

Now, we are ready to present the following main result. Let D_i be a block in BN (1) and $\{D\}_i = \{D_{i1}, ..., D_{iq_i}\}_i$ the corresponding block group in BN (3). If there exists a $D_{ij} \in \{D\}_i$, such that $Dim(C_i) = Dim(C_i)$ and

$$\delta_{cy_i} = \delta^2_{cu_i}; i = 1, 2, \ldots, k.$$

Then complete synchronization between BNs (1) and (2) occurs with respect to the DAs specified by D_i and D_{ij}. **Proof:** We prove that S_{ij} and D_{ij} in block group $\{D\}_i$ have one-to-one correspondence. That is, $S_{y_{ij}}$ of BN (3), $S_{y_{ij}} = S_i \times S_{ij}$, is a domain of BN (3).

We first prove $S_{y_{ij}}$ has only one attractor C_{y1} by contradiction. Suppose that $S_{y_{ij}}$ has two attractors C_{y1} and C_{y2} with cycle lengths k_{y1}, k_{y2}, respectively:

$$C_{y1} = \{p_1, p_2, \ldots, p_{k_{y1}}\}, \qquad C_{y2} = \{q_1, q_2, \ldots, q_{k_{y2}}\}.$$

We have

$$p_m = u_m x_m, \quad m = 1, 2, \ldots, k_{y1}, \qquad q_n = u_n x_n, \quad n = 1, 2, \ldots, k_{y2}.$$

Assume the cycle lengths in S_i and S_{ij} are k_u and k_x, respectively. Obviously, k_{y1} and k_{y2} are both the L.C.M. of k_u and k_x. Therefore, if $k_{y1} \neq k_{y2}$, then $k_{y1}/k_u \neq k_{y2}/k_u$, $k_x \neq k_x$, a contradiction occurs; if $k_{y1} = k_{y2}$, then

$k_{y1}/k_u = k_{y2}/k_u$, S_{ij} has two different attractors with equal length $C_{xi1} \neq C_{xi2}$, contradicting to that S_{ij} has only one attractor. Therefore, $S_{y_{ij}}$ has only one attractor.

Next, we prove there is no other state outside $S_i \times S_{ij}$ that converges to C_y by contradiction, assume that there exists a vector $\delta_0 \notin S_{y_{ij}}$ and $T_0 \in N$, such that $y(T_0, \delta_0) = C_y$ for $t > T_0$. We take $\delta_0 = u_0 x_0$. Since $\delta_0 \notin S_{y_{ij}}$, we have $u_0 \notin S_i$ or $x_0 \notin S_{ij}$. Suppose the first case holds, and then

$$u(t, u_0) = C_i \neq C_i,$$

where, C_i is the attractor in S_i. Therefore,

$$C_y = C_i C_{ij} \neq C_i C_{ij}.$$

This contradict to that a DA only has one attractor. Hence, there is no trajectory starting from outside of $S_{y_{ij}}$ converging to C_y. This proves that the input-determined DA in BN (2) has one-to-one correspondence with D_{ij} in $\{D\}_i$. Thus, we have $S_{y_{ij}} = S_i \times S_{ij}$.

Finally, we prove synchronization between BNs (1) and (2) occurs within the specified DAs. Since for any $u(0) \in S_i$ and $x(0) \in S_{ij}$, $y(0) = u(0)x(0)$. So $Col(P_1 y_0) \in Col(D_{ij})$. There exists a transition time $T(u(0), x(0)) > 0$, such that for $t > T(u(0), x(0))$, the state vector of $\tilde{y}$ takes values out of the column vectors in the position of permutation matrix C_{ij}. Therefore,

$$\delta_{cy_i} = \delta^2_{cu_i}, i = 1, 2, \ldots, k.$$

That is,

$$\delta_{cu_i} = \delta_{cx_i}, i = 1, 2, \ldots, k.$$

It means that once BNs (1) and (2) run into the cycles in D_i and D_{ij}, respectively, complete synchronization will take place within the specific DAs. Because the combinations of $u(0)$ and $x(0)$ are finite, we can always find such a T_p large enough for all combinations.

5 Cyclic patterns

Obviously, complete synchronization considered above is only one type of cyclic patterns between BNs (1) and (2) in CBNs. We can also consider other types. General cyclic pattern, in which we neither consider cycle length nor the node state correspondence between BNs (1) and (2). Rolling-Gear, a special type in general cyclic pattern, in which we consider only on cycle length relationship between (1) and (2). Rolling-Gear allows for a tiny cycle in a small BN to drive a large cycle in a BN of large-scale, which may reveal hidden order in life (Cheng et al. 2011a). General synchronization, a special type in Rolling-Gear, in which cycle length between BNs (1) and (2) is equal and node states have some special correspondence, such as complete synchronization, partial synchronization, anti-complete synchronization, synchronization on cycle length, etc. Similarly, for general synchronization, we can also establish corresponding criteria.

Observe that the discussion in last section relies on a particular combination of DAs in BNs (1) and (2). we can further extend our discussion to all possible combinations. Because the cyclic patterns in different combinations of DAs are independent, comparing to combination of S_i and S_{ij}, different type of cyclic pattern may occur within different combination of S_{i_1} and $S_{i_1 j_1}$, $i \neq i_1, j \neq j_1$; or even within S_i and S_{ij_1}. The initial states, $u(t_0)$ and $x(t_0)$, will jointly decide the type of cyclic pattern in the CBNs.

Next, we point out that $u(t_0)$ and $x(t_0)$ can have different effects on the occurrence of complete synchronization when it takes place within different combinations of DAs. Let D_i, $i = 1, \ldots, p$, be a block in (4) and $\{D\}_i = \{D_{i1}, \ldots, D_{iq_i}\}_i$ the corresponding block group in (5). We have the following results.

Corollary 1

1.1 If $q_i = 1$, $i = 1,\dots,p$, and there exists $i_1 \neq i_2$, such that complete synchronization of BNs (1) and (2) occurs within $D_{i_1 1}$ but does not within $D_{i_2 1}$, then complete synchronization depends only on $u(t_0)$ but not on $x(t_0)$.

1.2 If $q_i > 1$, $i = 1,\dots,p$, and there exists $i_1 \neq i_2$, $j_1 \neq j_2$, $j_1, j_2 = 1,\dots,q_i$, such that complete synchronization of BNs (1) and (2) occurs within $D_{i_1 j_1}$ but does not within $D_{i_2 j_2}$, then complete synchronization depends not only on $u(t_0)$ but also on $x(t_0)$.

1.3 If $q_i > 1$, $i = 1,\dots,p$, and there exists $j_1 \neq j_2$, $j_1, j_2 = 1,\dots,q_i$, for each $i = 1,\dots,p$, such that complete synchronization of BNs (1) and (2) occurs within $D_{i j_1}$ but does not within $D_{i j_2}$, then complete synchronization depends only on $x(t_0)$ but not on $u(t_0)$.

1.4 If for i, $i = 1,\dots,p$; j, $j = 1,\dots,q_i$, such that complete synchronization of BNs (1) and (2) occurs within all D_{ij}, then complete synchronization depends neither on $x(t_0)$ nor on $u(t_0)$.

The proof of above results is obvious. Particularly, for synchronization described in Corollary 1.4, we call it globally consistently complete synchronization, which means complete synchronization takes place for all combinations of DAs in CBNs.

6 Conclusion

We have studied synchronization of CBNs and established a algebraic criterion by using STP method. Based on this, we also considered the cyclic patterns in CBNs and found that the coupled system may exhibit various cyclic behaviours, depending on possible combinations of DAs of the BNs comprising the whole system.

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