Positive Invariance of Constrained Linear Continuous-Time Delay System with Delay Dependence

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***Abstract*-** In this paper, we apply the concept of positive invariance to continuous-time delay linear system. Necessary and sufficient algebraic conditions with delay dependence allowing to obtain the largest positively invariant set of delay system are given. The results can include information on the size of delay, and therefore, can be delay dependence positively invariant conditions. A numerical example is given to illustrate theoretical developments.

Keywords: Time delay systems; non-symmetrical constraints; polyhedral sets; positive invariance; asymptotic stability.

1   Introduction

Motivated by many practical problems, especially in the case of dynamic systems subject to constraints, the positively invariant sets have played a crucial role in synthesis and analysis of both linear discrete-time and continuous-time systems see (Khalil 2002, Blanchini 1999). Recently, it was shown that the positive invariance property can be exploited to obtain an explicit characterization of global switching behaviors of piecewise affine systems (Shen 2012). In the literature, there are several results involving application of the concept of positive invariance to linear discrete-time and continuous-time systems see (Blanchini 1999) and the references therein. To our knowledge, almost all these works are developed for dynamical systems which possesses no time delay, which is seldom the case in industrial applications. In particular, some delay-independent positively invariant set conditions have been proposed for linear delay system (Hmamed 2000, Hennet & Tarbouriech 1997, Dambrine, Richard & Borne 1995). Delay-dependent positively invariant set conditions for continuous-time and discrete-time delay system have also been developed respectively by (Dambrine, Goubet & Richard 1995) and (Hmamed 2000).

In the sequel, we will be concerned essentially with linear continuous-time delay system described by state space equation (2). We study the positive invariance property of the polyhedral set such that

$$D(\mathbb{I}_n, r_1, r_2) = \left\{ x \in \mathbb{R}^n \mid -r_2 \leq x \leq r_1 \right\} \tag{1}$$

with r_1 and r_2 being n dimensional real vectors with positive components.

The paper is structured as follows. In section II, we define the positive invariance and give some preliminary results. In section III, we derive necessary and sufficient algebraic conditions with delay dependence for polyhedral set $D(\mathbb{I}_n, r_1, r_2)$ to be positively invariant of linear continuous-time delay system.

Notation: If A denotes a matrix of $\mathbb{R}^{n \times n}$ and x, y vectors of $\mathbb{R}^n$ then: A^+ (respectively A^-) is the matrix whose components are given by $A_{ij}^+ = max(A_{ij}, 0)$ (respectively $A_{ij}^- = max(-A_{ij}, 0)$), A_1 (respectively A_2

) is the matrix whose components are given by A_{ii} if $i = j$ and A_{ij}^+ if $i \neq j$ (respectively 0 if $i = j$ and A_{ij}^- if $i \neq j$), and $|A|$ is the matrix whose components are given by $|A_{ij}|$, for $i, j = 1, \ldots, n$. $\mathbb{I}_n$ is the identity matrix $\mathbb{R}^{n \times n}$. $x \leq y$ if $x_i \leq y_i$, for $i = 1, \ldots, n$ and ∂D is the boundary of the set D.

2 Preliminary results

In this section, we present some definitions and useful results for the sequel. Consider the following linear continuous-time delay system

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - \tau), & \text{for } t > 0 \\ x(\theta) = \phi(\theta), & \text{for } \theta \in [-\tau, \, 0] \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$; $\tau > 0$ is the time delay and $A, A_d \in \mathbb{R}^{n \times n}$. It is assumed that for any continuous function ϕ bounded and measurable, the time delay system (2) has unique solution denoted by $x(t, \, \phi)$.

Definition 1. *A set D of $\mathbb{R}^n$ is said to be positively invariant for motions of system (2), if for every $\phi(s) \in D$ ($s \in [-\tau, \, 0]$), the motion $x(t, \, \phi) \in D$, for every $t > 0$.*

By using the Newton-Leibniz formula (Hale & Lunel 1993), we have

$$x(t - \tau) = x(t) - \int_{-\tau}^{0} \dot{x}(t + s) ds$$

$$= x(t) - \int_{-\tau}^{0} \left[Ax(t + s) + A_d x(t - \tau + s) \right] ds$$

This equation together (2), the original system can be rewritten as

$$\dot{x}(t) = (A + A_d)x(t) - A_d \int_{-\tau}^{0} \left[Ax(t + s) + A_d x(t - \tau + s) \right] ds \tag{3}$$

under arbitrary continuous initial condition

$$x(\theta) = \psi(\theta), \quad \theta \in [-2\tau, \, 0]. \tag{4}$$

It is noteworthy that the solution of the system with discrete delay (2) is also the solution of system with distributed delay (3–4), and the asymptotic stability of the system (3–4) can guarantee the asymptotic stability of original system (2), see (Hale & Lunel 1993) and (Niculescu 2001). For the sake of simplicity, the system with distributed delay is used to obtain the stability or, in the case of this paper, the positive invariant conditions for the system with discrete delay. The polyhedral set $D(\mathbb{I}_n, r_1, r_2)$ can be written as a set defined as

$$D(\mathbb{I}_{2n}, r) = \left\{ y \in \mathbb{R}^{2n} \mid y \leq r; \, r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \right\} \tag{5}$$

where $y(t) = \begin{pmatrix} x(t) \\ -x(t) \end{pmatrix}$. Therefore, the following proposition is straightforward.

Proposition 2. *The positive invariance of polyhedral set $D(\mathbb{I}_n, r_1, r_2)$ for system (3–4) is equivalent to the positive invariance of the set $D(\mathbb{I}_{2n}, r)$ for system defined by*

$$\dot{y}(t) = \widetilde{M}y(t) + \int_{-\tau}^{0} \left[\widetilde{Q}y(t + s) + \widetilde{R}y(t - \tau + s) \right] ds. \tag{6}$$

under arbitrary continuous initial condition

$$y(\theta) = \begin{pmatrix} \psi(\theta) \\ -\psi(\theta) \end{pmatrix}, \quad \theta \in [-2\tau, 0] \tag{7}$$

with

$$M = A + A_d \tag{8a}$$

$$Q = A_d A \tag{8b}$$

$$R = A_d^2 \tag{8c}$$

where $\widetilde{M}$, $\widetilde{Q}$ and $\widetilde{R}$ are such that

$$\widetilde{M} = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix}, \quad \widetilde{Q} = \begin{pmatrix} Q^- & Q^+ \\ Q^+ & Q^- \end{pmatrix} \text{ and } \widetilde{R} = \begin{pmatrix} R^- & R^+ \\ R^+ & R^- \end{pmatrix}. \tag{9}$$

Proof. By increasing the size of system (3–4) and observing that $M = M_1 - M_2$, $Q = Q^+ - Q^-$ and $R = R^+ - R^-$ the associated system to polyhedral set $D(\mathbb{I}_n, r_1, r_2)$ can be defined by (6–7). Since the set $D(\mathbb{I}_n, r_1, r_2)$ can be written as (5), then we obtain our result. $\square$

Remark 3. *The matrix $\widetilde{M}$ is Metzler, since its off diagonal components are nonnegative (Ait-Rami & Tadeo 2007), and both matrices $\widetilde{Q}$ and $\widetilde{R}$ are nonnegative (Haddad & Chelleboina 2005). In this case the system (6) can be seen as positive system for any nonnegative initial condition $0 \le y(\theta) \le r$, $\theta \in [-2\tau, 0]$ (Kaczorek 2006).*

3 Delay dependence positively invariant conditions

A necessary and sufficient condition for the non-symmetrical domain $D(\mathbb{I}_n, r_1, r_2)$ to be positively invariant for original system (2) with delay dependence is herein proposed.

Theorem 4. *The polyhedral set $D(\mathbb{I}_n, r_1, r_2)$ defined by (1) is a delay-dependent positively invariant set for system (2) if and only if the following condition holds*

$$\left(\widetilde{M} + \tau(\widetilde{Q} + \widetilde{R}) \right) r \le 0 \tag{10}$$

where $\widetilde{M}$, $\widetilde{Q}$, $\widetilde{R}$ are given in (9), and $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$

Proof. Necessity: We show this by contradiction. Suppose $D(\mathbb{I}_n, r_1, r_2)$ is positively invariant for system (2) but (10) does not hold, then there exists $i \in [1, n]$ such that

$$M_{ii} r_1^i + \sum_{\substack{j=1 \\ j \ne i}}^{n} \left(M_{ij}^+ r_1^j + M_{ij}^- r_2^j \right) + \tau \sum_{j=1}^{n} \left((Q_{ij}^- + R_{ij}^-) r_1^j + (Q_{ij}^+ + R_{ij}^+) r_2^j \right) > 0 \tag{11}$$

Consider the particular states $x_p(t)$, $x_p(t+s)$ and $x_p(t-\tau+s) \in \partial D(\mathbb{I}_n, r_1, r_2)$ $(-\tau \le s \le 0)$ such that

$$x_p(t) = \begin{cases} r_1^i, & \text{for } j = i \\ r_1^j, & \text{if } M_{ij} > 0 \text{ for } j \ne i \\ -r_2^j, & \text{if } M_{ij} < 0 \text{ for } j \ne i \\ 0, & \text{if } M_{ij} = 0 \end{cases}, \quad x_p(t+s) = \begin{cases} r_1^j, & \text{if } Q_{ij} < 0 \\ r_2^j, & \text{if } Q_{ij} > 0 \\ 0, & \text{if } Q_{ij} = 0 \end{cases} \quad j = 1, \ldots, m \quad \text{and}$$

$$x_p(t-\tau+s) \begin{cases} r_1^j, & \text{if } R_{ij} < 0 \\ r_2^j, & \text{if } R_{ij} > 0 \\ 0, & \text{if } R_{ij} = 0 \end{cases} \quad j = 1, \ldots, m \quad (12)$$

By using the system (3–4), the derivative of ith component of particular vector $x_p(t)$, $\forall t > 0$, can satisfy the following

$$\dot{x}_{p_i}(t) = M_{ii} r_1^i + \sum_{\substack{j=1 \\ j \ne i}}^{n} \left(M_{ij}^+ r_1^j + M_{ij}^- r_2^j \right) + \tau \sum_{j=1}^{n} \left((Q_{ij}^- + R_{ij}^-) r_1^j + (Q_{ij}^+ + R_{ij}^+) r_2^j \right) > 0$$

this implies that for all scalar $\varepsilon > 0$, $x_{p_i}(t+\varepsilon) > x_{p_i}(t)$. Hence $x_{p_i}(t+\varepsilon) \notin D(\mathbb{I}_n, r_1, r_2)$, a contradiction.
Sufficiency: It is shown in proposition 2, that system (6) is associated to set $D(\mathbb{I}_n, r_1, r_2)$. It follows from Lagrange's formula that the solution of (6) is given by

$$y(t) = e^{\widetilde{M}t} y(0) + \int_0^t e^{\widetilde{M}(t-\theta)} \left(\int_{-\tau}^0 \left(\widetilde{Q} y(\theta + s) + \widetilde{R} y(\theta + s - \tau) \right) ds \right) d\theta; \, t > 0.$$

where $\widetilde{M}$, $\widetilde{Q}$ and $\widetilde{R}$ are given by (9). If we consider $y(s) \le r$ for $s \in [-2\tau, 0]$, then in same way found in (El'sgol'ts & Norkin 1973) and (Hmamed, Benzaouia & Bensalah 1995), we obtain $y(t) \le r$ on the intervals $[0, 2\tau]$, $[2\tau, 4\tau], \ldots$ $\qquad\square$

If there exist a vector $\rho > 0$ such that $r_1 = r_2 = \rho$, then the asymmetric set $D(\mathbb{I}_n, r_1, r_2)$ can be transformed to symmetric set $D(\mathbb{I}_n, \rho, \rho)$ such that

$$D(\mathbb{I}_n, \rho, \rho) = \left\{ x \in \mathbb{R}^n \mid -\rho \le x \le \rho, \, \rho > 0 \right\}$$

Thus symmetrical case can be easily deduced and the following corollary is given by

Corollary 5. *The polyhedral set $D(\mathbb{I}_n, \rho, \rho)$ is a delay-dependent positively invariant set for system (2) if and only if the following condition hold*

$$\left(\overline{M} + \tau (|Q| + |R|) \right) \rho \le 0 \tag{13}$$

where $\overline{M}$ is the matrix whose components are given by M_{ii} if $i = j$ and $|M_{ij}|$ if $i \ne j$

Proof. By observing again that $\overline{M} = M_1 + M_2$, $|Q| = Q^+ + Q^-$, and $|R| = R^+ + R^-$, the condition (10) becomes the condition (13), proving our result. $\qquad\square$

Remark 6. *When delay time $\tau = 0$, the condition (10) becomes equal to $\widetilde{M} r \le 0$, where $M = A + A_d$. The result of positive invariance of the polyhedral set $D(\mathbb{I}_n, r_1, r_2)$ with respect to continuous system without delay is obviously obtained (Benzaouia & Hmamed 1993).*
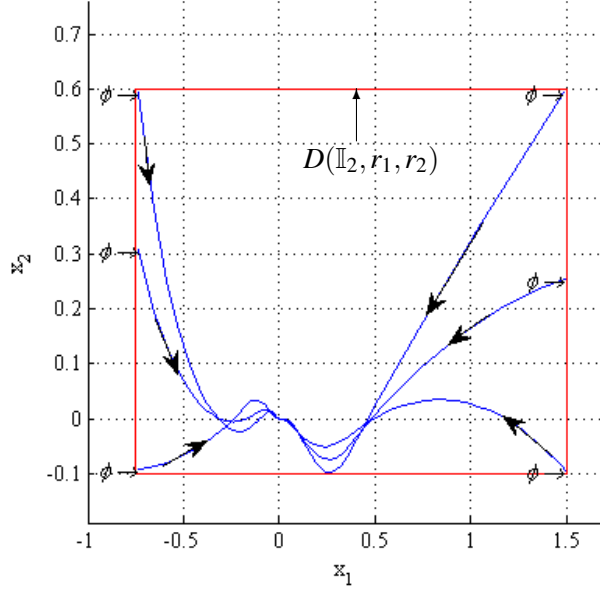
Fig. 1: The invariant polyhedron set for initial different states $\phi \in D(\mathbb{I}_2, r_1, r_2)$. The trajectories starting from set are given for $\tau = \tau_{\max} = 1.331$.

4 Example of rectangular constraint set

Consider the following dynamical system, for all positive times $(t > 0)$

$$\begin{cases} \dot{x}_1(t) = -x_1(t) - \dfrac{1}{2}x_2(t) \\ \dot{x}_2(t) = \dfrac{1}{2}x_1(t) - 2x_2(t) - \dfrac{1}{4}x_1(t-\tau) - \dfrac{1}{4}x_2(t-\tau) \end{cases} . \tag{14}$$

and assume that its state variables are subject to the constraints

$$-0.75 \le x_1 \le 1.5 \tag{15a}$$

$$-0.1 \le x_2 \le 0.6 \tag{15b}$$

Such constraints form a rectangle represented by the bounded polyhedron set $D(\mathbb{I}_2, r_1, r_2)$, where $r_1 = \begin{pmatrix} 1.5 \\ 0.6 \end{pmatrix}$ and $r_2 = \begin{pmatrix} 0.75 \\ 0.1 \end{pmatrix}$ as shown in figures 1 and 2. It is positively invariant of system (14) if the interval of delay τ fulfills the condition $\left(\widetilde{M} + \tau \left(\widetilde{Q} + \widetilde{R} \right) \right) r \le 0$. We find that $0 \le \tau \le 1.331$. If we take as initial states $\phi \in D(\mathbb{I}_2, r_1, r_2)$, the resultant trajectory (figure 1) does not violate the constraints. Alternatively, the trajectory of system (14) starting from $D(\mathbb{I}_2, r_1, r_2)$ for $\tau = 1.431 \notin [0\ 1.331]$, as show figure 2, violates the constraints. The set $D(\mathbb{I}_2, r_1, r_2)$ is not positively invariant with respect to system (14) for $\tau = 1.431$.

Obviously, we believe that these results find application to constrained regulator problem of linear continuous time delay systems while taking into account the delay parameter.

5 Conclusion

In this paper, the positive invariance of constrained continuous-time systems with delay is studied. Necessary and sufficient algebraic conditions with delay dependence for a given polyhedral set to be positively
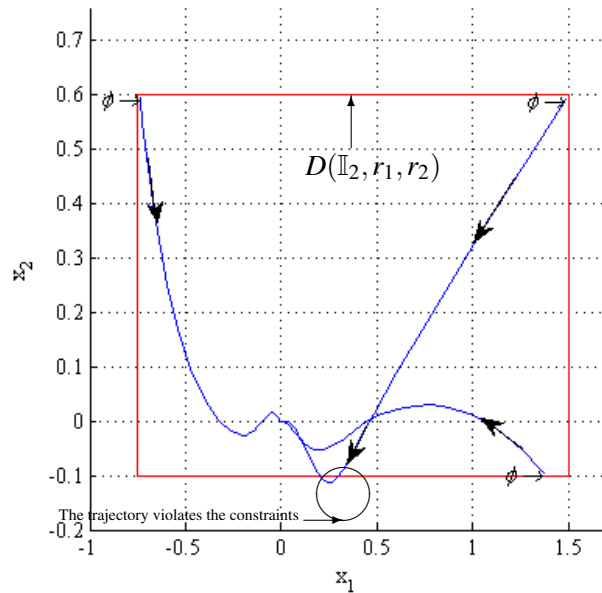
Fig. 2: $D(\mathbb{I}_2, r_1, r_2)$ is not positively invariant for delay $\tau = 1.431$. The trajectories starting from set are given for three initials conditions ϕ.

invariant with respect to motions of linear continuous-time delay system are established. The case of symmetrical constrained is easily obtained. The link of results given between systems with and without delay is established. Finally the delay dependent condition presented in this paper is shown to be a generalization of theorem 2-2 given in (Hmamed et al. 1995).

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