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The Analysis of the Non-linear Deflection of Non-straight Ludwick type Beams Using Lie Symmetry Groups

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Abstract - Nonlinear deflection of beams under the various forces and boundary conditions has been widely studied. The prediction of deflection of beams has been of great interest to generations of researchers. This seems to be a mundane problem as it is subject of textbooks on elementary mechanics of materials. Although both analytical and numerical solution have been found for specific type of loads, the general problem for beams that are not geometrically perfectly straight has not been approached so far of a systematic fashion. The present work presented a general method based on Lie symmetry groups that yields an exact solution to the general problem involving any arbitrary loading non-straight Ludwick type micro-beams. Lie symmetry method is used to reduce the order of the ODE describing the large deflection of the beam. The solution is validated against the particular cases of loading for which the large deflection problem has been solved and presented in the open literature. In this work an approach in solving the general problem based on Lie symmetry groups and a general analytical solution of the problem is presented below. The objective of the present work is to investigate a versatile mathematical method into the deflection of geometrically non-straight cantilever beams subjected point loads and moments applied at the free end while experiencing non-linear deflection. Lie symmetry method presented below can be used to any geometry of the bended beams under to condition that there is no residual stress in the unloaded beam. The deflection equation was calculated based on Ludwick experimental strain-stress curve. The integral equation was solved numerically and the end beam deflection and rotation were calculated. The same problem of large deflection cantilever beams made from materials behaving of non-linear fashion under the tip point force was solved by finite difference methods.

Keywords: Non-Linear Deflection, Ludwick Beams, Lie Symmetry

1. Introduction

Nonlinear deflection of beams under the various forces and boundary conditions has been widely studied. The prediction of deflection of beams has been of great interest to generations of researchers. This seems to be a mundane problem as it is subject of textbooks on elementary mechanics of materials. Although both analytical and numerical solution have been found for specific type of loads, the general problem for beams that are not geometrically perfectly straight has not been approached so far of a systematic fashion. Despite the interest in the subject, so far there is no general solution to describe the general case of loading. This research presents an approach in solving the general problem based on Lie symmetry groups and a general analytical solution of the problem is presented below. The objective of the present work is to investigate a versatile mathematical method into the deflection of geometrically non-straight cantilever beams subjected point loads and moments applied at the free end while experiencing non-linear deflection for Ludwick type Beams. Lie symmetry method presented below can be used to any geometry of the bended beams under to condition that there is no residual stress in the unloaded beam.

Nonlinear deflection of beams subjected to various types of forces and boundary conditions have been extensively studied. The differential equation of large deflection of cantilever beam under a point force at the tip was solved in 1945

[1]. In that approach the differential equation of the slope of the beam versus the length of the deflected curve was formulated and solved based on complete second and first kind elliptic integrals. The differential equation of slope versus length of the deflected curve based on consideration of shear force was numerically solved [2]. The authors used finite difference methods to solve ordinary differential equation (ODE) for distributed force on cantilever and simple supported beams. They also used the same method to solve the ODE of the simple supported beam under a point force. A numerical solution for the tapered cantilever beam under a point force at tip was presented in 1968 [3]. The author converted ODE to non-dimensional ODE and used a computer to solve it. A cantilever beam, made from materials exhibiting nonlinear properties subjected to a point force was also studied [4].

The deflection equation was calculated based on Ludwick experimental strain-stress curve. The integral equation was solved numerically and the end beam deflection and rotation were calculated. The same problem of large deflection cantilever beams made from materials behaving of non-linear fashion under the tip point force was solved by finite difference methods [5].

The authors solved the nonlinear ODE of curvature for a cantilever made from nonlinear characteristics a material and subjected to point force at the tip by numerical methods. Power series and neural network were used to solve large deflection of a cantilever beam under tip force [6]. Nonlinear ODE were decomposed to a system of ODEs and solved by neural networks. Large deflections of cantilever beams made from nonlinear elastic materials under uniform distributed forces and a point force at tip were also studied [7]. In this work a system of nonlinear ODEs was developed to model the system which was further solved by Runge-Kutta method. Researchers [8] in used almost a similar method that was used in [1] to solve the large deflection of a cantilever under the point force at the tip and they validated their results with experiments. Also they used non-dimensional formulation to simplify the nonlinear deflection to linear analysis. They showed that nonlinear small deflection is same as those found through the linear analysis. Two dimensional loading of cantilever beams with point forces loads at the free end was studied for non-prismatic and prismatic beams [9]. Authors formulated the model for the general loading conditions in beams. The result is a nonlinear PDE which is presented in this paper. Further, the authors numerically solved the non-dimensional equation using a polynomial to define the rotating angle of the beam. They presented some examples applied to their methods. A cantilever beam subjected to a tip moment with nonlinear bimorph material was theoretically and numerically studied [10]. The authors used an exact solution for the deflection of a cantilever with a moment applied at the tip. Cantilever beam under uniform and tip point force was numerically and experimentally studied [11]. In this study, the authors used a system of ODEs to solve numerically this problem. Finite difference methods for analysis of large deflection of a non-prismatic cantilever beam subjected to different types of continuous and discontinuous loadings was studied [12]. Authors formulated the problem based on [9] and further used quasi-linearization central finite differences method to solve the problem. An explicit solution for large deflection of cantilever beams subjected to point force at the tip was obtained by using the homotopy analysis method (HAM) presented in [13]. Large deflection of a non-uniform spring-hinged cantilever beam under a follower point force at the tip was formulated and solved numerically [14].

2. Ludwick Materials

Generally the relation between stress and strain under small stress conditions is linear as stated by Hook Law. By increasing stress nonlinear behavior of material is expressed. For most of materials the non-linear behaviour can be defined as a stress-strain relation like:

$$\sigma = \mathrm{E}\varepsilon^{\frac{1}{n}}$$

$$\mathrm{I_n} = \int y^{\frac{n+1}{n}} dA = \left(\frac{1}{2}\right)^{\frac{n+1}{n}} \left(\frac{n}{2n+1}\right) bh^{\frac{2n+1}{n}}$$

This kind of materials are called Ludwick materials. Here, *B* and *n* represent constants related to material properties. For a rectangular cross section of beam subjected to a moment at the free end.

2. Formulation of the Large Deflection of Not-Straight Beam Problem

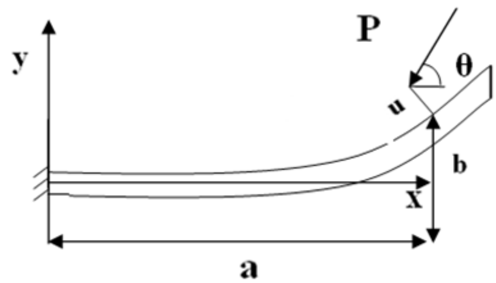

Fig. 1: A non-straight cantilever under tip forces and moment.

A non-straight cantilever beam as shown in figure 1 is subjected a point forces. Internal reaction moment in any cross section on the left hand side of cantilever can be written as:

$$M(x,y) = V(x_0 - x) + H(y_0 - y) + M_0 \tag{1}$$

where:

$$V = P\sin(\theta_0)$$
$$H = P\cos(\theta_0)$$
$$M_0 = P.u$$

u is distance from end of tip to neutral axis of cantilever beam.

Euler–Bernoulli moment–curvature relationship gives:

$$\frac{d\theta}{d\rho} = \frac{M(x,y)}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}} \tag{2}$$

Where:

$$\frac{dx}{d\rho} = \cos(\theta) \tag{3}$$
$$\frac{dy}{d\rho} = \sin(\theta) \tag{4}$$

θ is angle of curved beam.

Derivative of (2) gives:

$$\frac{d^2\theta}{d\rho^2} = -\frac{V\cos(\theta) + H\sin(\theta)}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}} \tag{5}$$

The boundary conditions for this ODE are:

$$\theta|_{\rho=0} = 0 \tag{6}$$

$$\left.\frac{d\theta}{d\rho}\right|_{\rho=L} = \frac{M_0}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}} \tag{7}$$

L is length of cantilever.
The equation (5) can be written as

$$\frac{d^2\theta}{d\rho^2} = \frac{-V\cos(\theta) - H\sin(\theta)}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}} \tag{8}$$

One can show that for a second order ODE infinitesimal transformation $\eta^{(2)}$ can be calculated as in [12, 13]

$$\begin{aligned}\eta^{(2)} &= D^{(2)}\eta - \sum_{j=1}^{2}\frac{2}{(2-j)!\,j!}\theta_{2-j+1}D^{j}\xi \\ &= \eta_{\rho\rho} + (2\eta_{\rho\theta} - \xi_{\rho\rho})\dot{\theta} + (\eta_{\theta\theta} - 2\xi_{\rho\theta})\dot{\theta}^2 - 2\xi_{\theta\theta}\dot{\theta}^3 + (\eta_\theta - 2\xi_\rho - 3\xi_\theta\dot{\theta})\ddot{\theta} \\ &= \xi\omega_p + \eta\omega_\theta + (\eta_\theta - \xi_\rho)\dot{\theta} - \xi_\theta\dot{\theta}^2)\omega_\theta\end{aligned} \tag{9}$$

Where:

$$\omega = -\frac{V\cos(\theta) + H\sin(\theta)}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}$$

By decomposing (9) into a system of PDEs, ξ and η can be calculated. Most of Lie symmetries including rotation translation and scaling could be found with the help of the below transformations:

$$\begin{aligned}\xi &= C_1 + C_2\,\rho + C_3\theta \\ \eta &= C_4 + C_5\,\rho + C_6\theta\end{aligned} \tag{10}$$

where:
$C_1, C_2, C_3, C_4, C_5, C_6$ are constant numbers.
Substitution (9) in (10) yields:

$$(C_6 - 2C_2 - 3C_3\dot{\theta})\left(-\frac{V\cos(\theta) + H\sin(\theta)}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}\right) = \left(\frac{V\cos(\theta) - H\sin(\theta)}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}\right)(C_4 - C_2\rho + C_6\dot{\theta}) \tag{11}$$

By comparing terms, one can show that

$$C_2 = C_3 = C_4 = C_5 = C_6 = 0 \tag{12}$$

Therefore:

$$\begin{aligned} \xi &= C_1 \\ \eta &= 0 \end{aligned} \tag{13}$$

Equations (13) give the general transformation for (8). It is possible to consider ξ and η as:

$$\xi(\rho, \theta) = 1 \tag{14}$$
$$\eta(\rho, \theta) = 0 \tag{15}$$

Hence, $X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$, is called a point symmetry, can be simplified as an operator:

$$X = \frac{\partial}{\partial \rho} \tag{16}$$

canonical coordinates, $s(\rho, \theta)$ and $r(\rho, \theta)$ can be derived from the solution of below ODE for $r(\rho, \theta)$:

$$\frac{d\theta}{d\rho} = \frac{\eta(\rho, \theta)}{\xi(\rho, \theta)} \tag{17}$$

and $s(\rho, \theta)$ will be:

$$s(\rho, \theta) = \left(\int \frac{d\rho}{\xi(\rho, \theta(r, \rho))}\right)\Big|_{r=r(\rho,\theta)} \tag{18}$$

Substituting (14) and (15) in (17) and (18) gives

$$r(\rho, \theta) = \theta \tag{19}$$
$$s(\rho, \theta) = \rho \tag{20}$$

By defining $u(r)$ as:

$$u(r) = \frac{1}{\frac{d\theta}{d\rho}} \tag{21}$$

One can show that:

$$\frac{du(r)}{dr} = -\frac{\frac{d^2\theta}{d\rho^2}}{\left(\frac{d\theta}{d\rho}\right)^2} \tag{22}$$

So:

$$\frac{d^2\theta}{d\rho^2} = -u(r)^{-3}\frac{du(r)}{dr} \tag{23}$$

Substitution of (21) and (22) in (23) gives:

$$\frac{du(r)}{dr} = \left(\frac{V\cos(\theta) + H\sin(\theta)}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}\right)u(r)^3 \tag{24}$$

Substitution of (19) and (20) in (24) will yield:

$$\frac{du(\theta)}{d\theta} = \left(\frac{V\cos(\theta) + H\sin(\theta)}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}\right)u(\theta)^3 \tag{25}$$

For a first order ODE like (25) $\eta^{(1)}$ can be written as:

$$\eta_\theta + (\eta_u - \xi_\theta)\omega - \xi_u\omega^2 = \xi\omega_\theta + \eta\omega_u \tag{26}$$

Where:

$$\omega = \left(\frac{V\cos(\theta) + H\sin(\theta)}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}\right)u(\theta)^3$$

Substituting ω and its derivatives in (26) yields

$$\begin{aligned}\eta_\theta &+ \frac{1}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}(\eta_u - \xi_\theta)\big(V\cos(\theta) + H\sin(\theta)\big)u^3 - \left(\frac{1}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}\right)^2 \xi_u\big(V\cos(\theta) + H\sin(\theta)\big)^2u^6 = \\ &\frac{1}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}\xi\big(-V\cos(\theta) + H\sin(\theta)\big)u^3 + \frac{30}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}\eta\big(V\cos(\theta) + H\sin(\theta)\big)u^2\end{aligned} \tag{27}$$

From (27), equating u^6 and free term, it is found that

$$\eta_\theta = \xi_u = 0 \tag{28}$$

By comparing the coefficients of u^3, one can write

$$(\eta_u - \xi_\theta)\big(V\cos(\theta) + H\sin(\theta)\big) = \xi\big(-V\cos(\theta) + H\sin(\theta)\big) \tag{29}$$

To satisfy (29), ξ must be assumed as zero

$$\xi = 0 \tag{30}$$

hence (27) becomes

$$\eta_u u = 3\eta \tag{31}$$

The integral of this ODE is:

$$\eta = u^3 \tag{32}$$

So $X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$ becomes:

$$X = u^3\frac{\partial}{\partial u} \tag{33}$$

Based on (33), canonical coordinates can be calculated from (17) and (18) as:

$$r(\theta, u) = \theta \tag{34}$$

$$s(\theta, u) = \frac{1}{2u^2} \tag{35}$$

Substituting (34) and (35) in $\frac{ds}{dr} = \frac{s_x + s_y f(x,y)}{r_x + r_y f(x,y)} = F(r)$ [13]yields:

$$\frac{ds}{dr} = \left(\frac{V\cos(r) + H\sin(r)}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}\right) \tag{36}$$

So:

$$S = \left(\frac{V\cos(r) + H\sin(r)}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}\right) + C_1 \tag{37}$$

Further, by substituting (21), (34) and (35) in (37) one gets:

$$\left(\frac{d\theta}{d\rho}\right)^2 = \left(\frac{2}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}\right)\left(-V\cos(r) + H\sin(r)\right) + C_1 \tag{38}$$

By considering boundary condition (6), C_1 becomes:

$$C_1 = \left(\frac{M_0}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}\right)^2 - \left(\frac{2}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}\right)\left(-V\cos(\theta_f) + H\sin(\theta_f)\right) \tag{39}$$

where θ_f is final angle in the tip.
Solution of (38) yields:

$$\rho = \int \frac{d\theta}{\sqrt{\left(\frac{2}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}\right)\left(-V\cos(\theta_f) + H\sin(\theta_f)\right) + C_1}} + C_2 \tag{40}$$

From equations (3) and (4) one can write equation (38) as:

$$\int_0^a dx = \int_0^{\theta_f} \frac{d\theta\cos(\theta)}{\sqrt{\left(\frac{2}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}\right)\left(-V\cos(\theta_f) + H\sin(\theta_f)\right) + C_1}} \tag{41}$$

$$\int_0^b dy = \int_0^{\theta_f} \frac{d\theta\sin(\theta)}{\sqrt{\left(\frac{2}{E\left(\frac{1}{2}\right)^{\frac{n+1}{n}}\left(\frac{n}{2n+1}\right)bh^{\frac{2n+1}{n}}}\right)\left(-V\cos(\theta_f) + H\sin(\theta_f)\right) + C_1}} \tag{42}$$

which relationship, after the integration will yield the deflection at the tip of the not-straight cantilever beam.

4. Conclusion

The deflection of geometrically non-straight cantilever beams subjected point loads and moments applied at the free end while experiencing non-linear deflection for Ludwick type Beams was obtained based on Lie symmetry method to reduce the order of the ODE describing the large deflection of the beam. Hereby general analytical solution of the deflection for the not-straight beam was systematically presented. This method is very general and powerful method for solving nonlinear differential equations.

References:

[1] K. E. Bisshopp and D. Drucker, "Large deflection of cantilever beam," *Notes*, vol.101, no. 3.

[2] T. M. Wang, S. L. Lee, and O. C. Zienkiewicz, "A numerical analysis of large deflections of beams," *International Journal of Mechanical Sciences*, vol. 3, no. 3, pp. 219-228, 1961.

[3] J. D. Kemper, "Large deflections of tapered cantilever beams," *International Journal of Mechanical Sciences*, vol. 10 no. 6, pp. 469-478, 1968.

[4] G. Lewis and F. Monasa, "Large deflections of cantilever beams of non-linear materials of the Ludwick type subjected to an end moment," *International Journal of Non-Linear Mechanics*, vol. 17, no. 1, pp. 1-6, 1982.

[5] F. Monasa and G. Lewis, "Large deflections of point loaded cantilevers with nonlinear behaviour," *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, vol. 34, no. 1, pp. 124-130, 1983.

[6] M. H. Ang Jr, W. Wei, and L. Teck-Seng, *On the estimation of the large deflection of a cantilever beam*, 1993.

[7] K. Lee, "Large deflections of cantilever beams of non-linear elastic material under a combined loading," *International Journal of Non-Linear Mechanics*, vol. 37, no. 3, pp. 439-443, 2002.

[8] T. Beléndez, C. Neipp, and A. Beléndez, "Large and small deflections of a cantilever beam," *European journal of physics*, vol. 23, pp. 371, 2002.

[9] M. Dado and S. Al-Sadder, "A new technique for large deflection analysis of non-prismatic cantilever beam," *Mechanics Research Communications*, vol. 32, no. 6, pp. 692-703, 2005.

[10] C. Baykara, U. Guven, and I. Bayer, "Large deflections of a cantilever beam of nonlinear bimodulus material subjected to an end moment," *Journal of Reinforced Plastics and Composites*, vol. 24, no. 12, pp. 1321-1326, 2005.

[11] T. Belendez, et al., "Numerical and experimental analysis of large deflections of cantilever beams under a combined load," *Physica scripta*, pp. 61, 2005.

[12] S. Al-Sadder and R. A. O. Al-Rawi, “Finite difference scheme for large-deflection analysis of non-prismatic cantilever beams subjected to different types of continuous and discontinuous loadings,” *Archive of Applied Mechanics*, vol. 75, no. 8, pp. 459-473, 2006.
[13] J. Wang, J. K. Chen, and S. Liao, “An explicit solution of the large deformation of a cantilever beam under point load at the free tip” *Journal of Computational and Applied Mathematics*, vol. 212, no. 2, pp. 320-330, 2008.
[14] B. S. Shvartsman, “Large deflections of a cantilever beam subjected to a follower force,” *Journal of Sound and Vibration*, vol. 304, no. 3-5, pp. 969-973, 2007.