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Viscosity Measurement for Picoliter Liquid Volumes Using Twin Micro-Cantilever Beams

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Abstract – In this paper a novel method to measure viscosity of a liquid with a volume of fraction of pico-litre is presented. The measurement of viscosity keeps into account the effect of evaporation which can result in the total evaporation of the liquid in a short time. At the present time there are several standard methods to measure viscosity of a fluid. All methods require larger volume of liquid and it takes rather long time to measure viscosity. No effect of evaporation is considered in these methods given the large volume of fluid that is used. In the proposed method viscosity of a droplet is measured by considering that in ambient condition such small droplet evaporates in less than a second. The principle of measurement includes two parallel micro-cantilever beams for which their individual dynamic response could be measured. Micro-cantilever beams are being used in MEMS mostly as inertial sensors. In this research the small volume of liquid is assumed to be positioned between the two parallel micro-cantilever beams. The two micro-beams are assumed to be respond to the dynamic excitation with a non-linear response which will be proved below that is related to the viscosity of the fluid. The dynamic system was modelled as a discreet three degree of freedom mass-damping-spring. The nonlinear differential equations governing the performance of the above mentioned system is further solved and the results are analysed. The initial deflection of micro-cantilever beams due to capillary force yields to the dynamic behaviour of micro-cantilever beams expressed in three differential equations that are nonlinear and stiff. The used algorithm to solve the coupled differential equations is called LSODE and in usually capable to perform the computation solution for the stiff system of nonlinear initial value differential equations. The time response and phase diagram of the deflection of the beams for the mass-spring-damper system are numerically derived and the results are compared for two circumstances: the liquid would not evaporate and the liquid evaporates. Conclusion with regards to the equipment capable to perform such measurements are drawn.

***Keywords*:** MEMS, Viscosity Measuring, Twin Micro-Cantilever Beams, Nonlinear Dynamics.

1. Introduction

The French physicist Jean Léonard Marie Poiseuille, derived for first time an equation called Poiseuille's law to calculate the viscosity of a liquid [1]. After him several methods were developed to measure viscosity of liquids [2-8]. Although very accurate ways were developed and are presently used to measure the viscosity, still the new technics are developing with specific applications [9-13]. In this investigation, a new method was presented to measure viscosity of liquid in pico-litre scale. The present problem raises two challenges which are: the reduced volume of liquid and the evaporation of the small volume of liquid. To overcome these issues, the vibration of two parallel micro-cantilever beams was assumed. The dynamics of twin micro-cantilever beams system was studied for first time on [14]. In this investigation the response of two symmetric micro-cantilever beams was investigated by analysing the vibrating response while subjected to a difference of potential close to the pull-in value. The system of two ODEs (Ordinary differential equations) was reduced to one nonlinear ODE. Further, Lie symmetry method was used to reduce the order of the differential equation. The pull-in voltage of the system was numerically calculated from the first order ODE and this way the behaviour of the system close to pull in voltage was studied. Twin micro-cantilever beams with different dimensions was studied in [15]. In this research behaviour of two micro-cantilever beams with different thickness under pull-in voltage was studied. The governed system of nonlinear ordinary differential equations was derived. To solve numerically the system of stiff nonlinear ODEs an advanced algorithm called LSODE in Maple was used. The sensitivity of the system was studied under conditions close to pull-in voltage. The results of solving a system of ODEs was compared with single ODE [14] when thickness of beams are identical. The effect of liquid between two beams was studied in [16]. The authors used the same concepts in [15] and studied the behaviour of two micro-cantilever beams under pull-in voltage condition and liquid droplet in between.

2. Nomenclature

The following variables are used in the modeling of phenomena:

$x(t)$ Deflection of the beam	ν Poisson ratio
$\dot{x}$ Velocity	ξ Damping factor
$\ddot{x}$ Acceleration	A Area of cross section on the beam
k Spring constant	m Mass of the beam
C Damping	ω_n Natural frequency
L Length of the beams	g Initial distance between the beam and substrate
h Height of the beams	b Width of the beams
E Young's modulus	ρ Density

3. The Mathematical Model

A dynamic model generally accepted to simulate of the vibration of micro-cantilever beams is the discrete lumped-mass system – with one degree of freedom. The equation describing the dynamics of a micro-cantilever beam can be expressed as the below equation (1):

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0 \tag{1}$$

Where

$$\ddot{x} = \frac{d^2x(t)}{dt^2}, \qquad \dot{x} = \frac{dx(t)}{dt}, \qquad x = x(t)$$

Figure 1 represents schematics the dynamic model of a symmetric set of twin of micro-cantilever beams with a droplet between them by using an equivalent mass-spring-damper system:

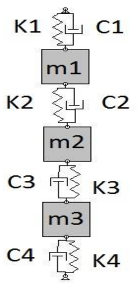

Fig. 1: The schematic view of a mass-spring-damper system of twin micro-cantilever beams with a droplet between them.

In this simulation based on Tylor-Analogy break-up model [16] the droplet was considered as a mass-damper-spring. The external force, damping and stiffness of the droplet can be considered as:

$$\frac{F}{m} = C_F\frac{\rho_g u_{rel}^2}{\rho_l r}, \qquad 2\xi\omega_n = C_d\frac{\mu_l}{\rho_l r^2}, \qquad \omega_n^2 = C_k\frac{\sigma}{\rho_l r^3} \tag{2}$$

Where:
ρ_g is gas density
ρ_l is liquid density
r is radius of droplet when it is in a spherical shape
C_F, C_d, C_k are model constants

The dynamic system in figure 1 can be described as a set of nonlinear stiff differential equations:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{x_1}(t) \\ \ddot{x_2}(t) \\ \ddot{x_3}(t) \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix}\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ \dot{x_3}(t) \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

The equation (3) can be expanded into the following equations:

$$m_1 \frac{d^2x_1(t)}{dt^2} + (c_1 + c_2)\frac{dx_1(t)}{dt} + (k_1 + k_2)x_1(t) - c_2 \frac{dx_2(t)}{dt} - k_2x_2(t) = 0 \tag{4}$$

$$m_2 \frac{d^2x_2(t)}{dt^2} + (c_2 + c_3)\frac{dx_2(t)}{dt} + (k_2 + k_3)x_2(t) - c_2 \frac{dx_1(t)}{dt} - k_2x_1(t) - c_3 \frac{dx_3(t)}{dt} - k_3x_3(t) = 0 \tag{5}$$

$$m_3 \frac{d^2x_3(t)}{dt^2} + (c_3 + c_4)\frac{dx_3(t)}{dt} + (k_3 + k_4)x_3(t) - c_3 \frac{dx_2(t)}{dt} - k_3x_2(t) = 0 \tag{6}$$

Equation (4) is assigned to the upper beam, equation (6) is considered for the lower beam while equation (5) is written for the droplet. The initial conditions for the system of equations (3) are defined by the initial deflection and speed at reference position and time as following:

$$x_{|_{t=0}} = x_0 \text{ and } \dot{x}_{|_{t=0}} = 0 \tag{7}$$

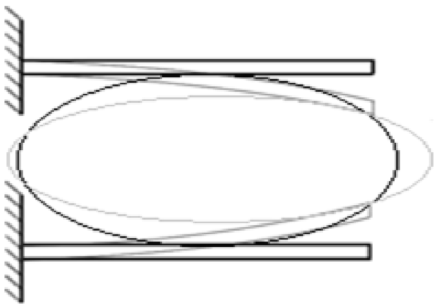

Fig. 2: Two set micro-cantilever beams with a droplet between them.

To simulate the evaporation of the droplet, an exponential evaporation law was considered and related to the mass, stiffness and damping of the droplet. In this first approach the exponential constant numbers for mass, stiffness and damping are considered same. The following values should be considered in the equation (3):

$$m = m_0e^{-\lambda t}, k = k_0e^{-\lambda t}, c = c_0e^{-\lambda t} \tag{8}$$

The equation (3) will become:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_{02}e^{-\lambda t} & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{x_1}(t) \\ \ddot{x_2}(t) \\ \ddot{x_3}(t) \end{bmatrix} + \begin{bmatrix} c_1 + c_{02}e^{-\lambda t} & -c_{02}e^{-\lambda t} & 0 \\ -c_{02}e^{-\lambda t} & (c_{02} + c_{03})e^{-\lambda t} & -c_{03}e^{-\lambda t} \\ 0 & -c_{03}e^{-\lambda t} & c_{03}e^{-\lambda t} + c_4 \end{bmatrix}\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ \dot{x_3}(t) \end{bmatrix} + \begin{bmatrix} k_1 + k_{02}e^{-\lambda t} & -k_{02}e^{-\lambda t} & 0 \\ -k_{02}e^{-\lambda t} & (k_{02} + k_{03})e^{-\lambda t} & -k_{03}e^{-\lambda t} \\ 0 & -k_{03}e^{-\lambda t} & k_{03}e^{-\lambda t} + k_4 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

The equations (4), (5) and (6) will be as following:

$$m_1\frac{d^2x_1(t)}{dt^2}+\left(c_1+c_{02}e^{-\lambda t}\right)\frac{dx_1(t)}{dt}+\left(k_1+k_{02}e^{-\lambda t}\right)x_1(t)-c_{02}e^{-\lambda t}\frac{dx_2(t)}{dt}-k_{02}e^{-\lambda t}x_2(t)=0 \tag{10}$$

$$m_{02}e^{-\lambda t}\frac{d^2x_2(t)}{dt^2}+(c_{02}+c_{03})e^{-\lambda t}\frac{dx_2(t)}{dt}+(k_{02}+k_{03})e^{-\lambda t}x_2(t)-c_{02}e^{-\lambda t}\frac{dx_1(t)}{dt}-k_{02}e^{-\lambda t}x_1(t)-c_{03}e^{-\lambda t}\frac{dx_3(t)}{dt}-k_{03}e^{-\lambda t}x_3(t)=0 \tag{11}$$

$$m_3\frac{d^2x_3(t)}{dt^2}+\left(c_{03}e^{-\lambda t}+c_4\right)\frac{dx_3(t)}{dt}+\left(k_{03}e^{-\lambda t}+k_4\right)x_3(t)-c_{03}e^{-\lambda t}\frac{dx_2(t)}{dt}-k_{03}e^{-\lambda t}x_2(t)=0 \tag{12}$$

Two symmetric parallel micro-cantilever beams are assumed to have same dimensions, just the thickness of one beam is assumed as different than the other one. The micro-beams are assumed being made from same material while the droplet is positioned between them. It implies that $L_1=L_2, b_1=b_2$ and $E_1=E_2$. Because of the small and large coefficients of the system of ODEs, the system will be a stiff system of nonlinear ODEs. A numerical approach is inevitable. One of efficient ways to solve a stiff system of ODEs is using mathematical software. In the current work Maple was used. Although there are several different methods for solving the system of stiff ODEs in Maple, the authors found out that the best algorithm for nonlinear stiff system of ODEs is LSODE algorithm. The LSODE method is used to solve stiff ODE with initial value. The numerical analysis was developed for the system of stiff nonlinear ODEs. The numerical results are presented in the following section.
The properties of the two beams are assumed as below:

$$\begin{gathered}L_1=L_2=200\ [\mu m]\\ b_1=b_2=20\ [\mu m]\\ h_1=2\ [\mu m]\\ h_2=5[\mu m]\\ E_1=E_2=169\ [GPa]\\ g=10\ [\mu m]\\ \xi_1=\xi_2=0.1\\ \rho_1=\rho_2=2300\ [kg/m^3]\\ \nu_1=\nu_2=0.3\\ K=\frac{2Ebh^2}{3L^3}\\ \omega_n=\sqrt{\frac{K}{m}}\\ A_1=A_2\ \text{(the modified area [17])}\end{gathered} \tag{13}$$

4. The Numerical Analysis Of The Model

The numerical solution of the nonlinear system of stiff ODEs in (9) with the values as set in (13) will be presented in this section. Given that the two micro-cantilever beams have different thickness, the dynamic response of the two beams is different. The thicker beam is significantly less sensitive and thus, the response of that would not be presented in this paper. The following graphs are two sets of plots for time response and phase diagram of micro-cantilever beam and the droplet. Figure 3 shows time response of the thinner beam and droplet with a nominal viscosity (of water) under two initial deflections. Figure 4 shows the phase diagram of the same beam for same two initial deflections. As one could see, the response of the micro-cantilever beam does not show sensitivity to the viscosity of the droplet. Figures 5 and 6 show the time response and phase diagram of droplet under 1[μm] initial deflection of the micro-cantilever beam for two very different liquid viscosities. Same for figures 7 and 8 which show the time response and phase diagram of droplet under 2 [μm] initial deflection of micro-cantilever beam and same two different viscosities of the liquid droplet.

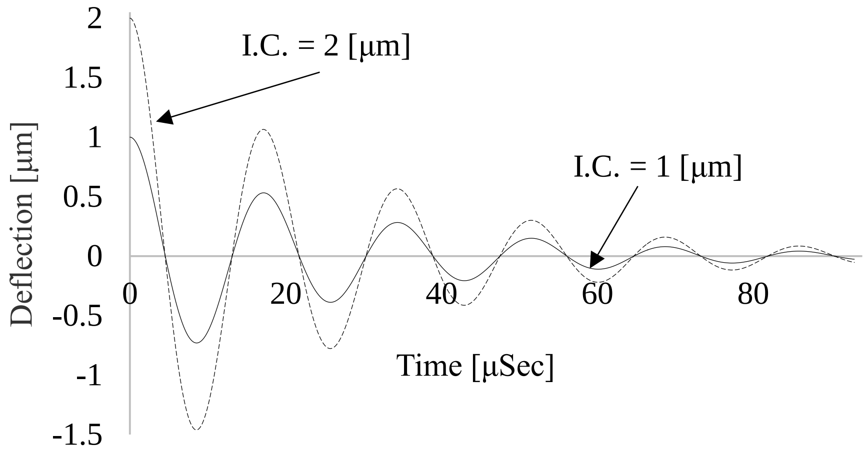

Fig. 3: Response of the thin micro-cantilever beam for two different initial deflection (1 [μm] and 2 [μm], any viscosity).

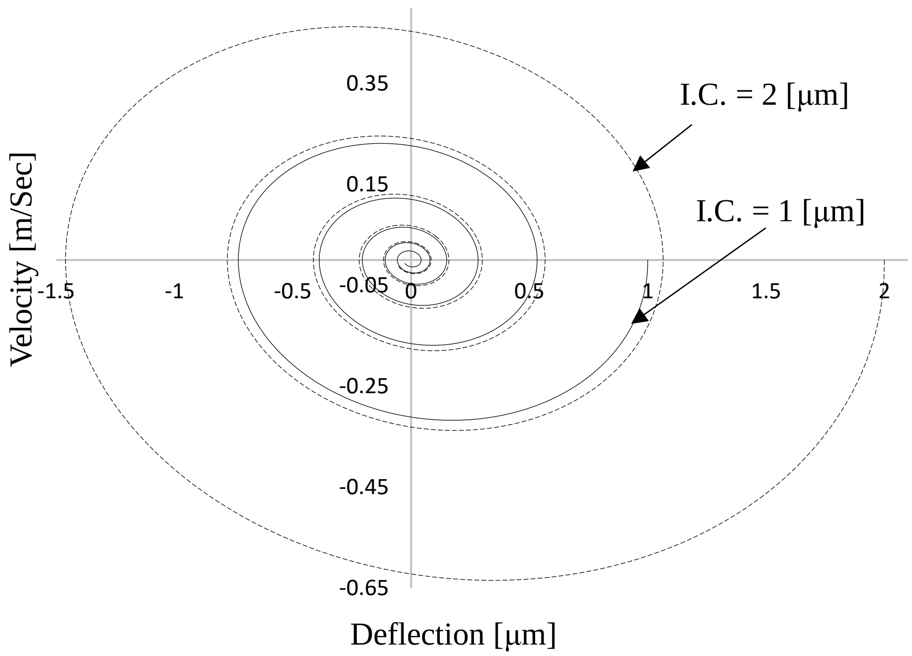

Fig. 4: Phase diagram of micro-cantilever beam with different initial deflection (any viscosity).

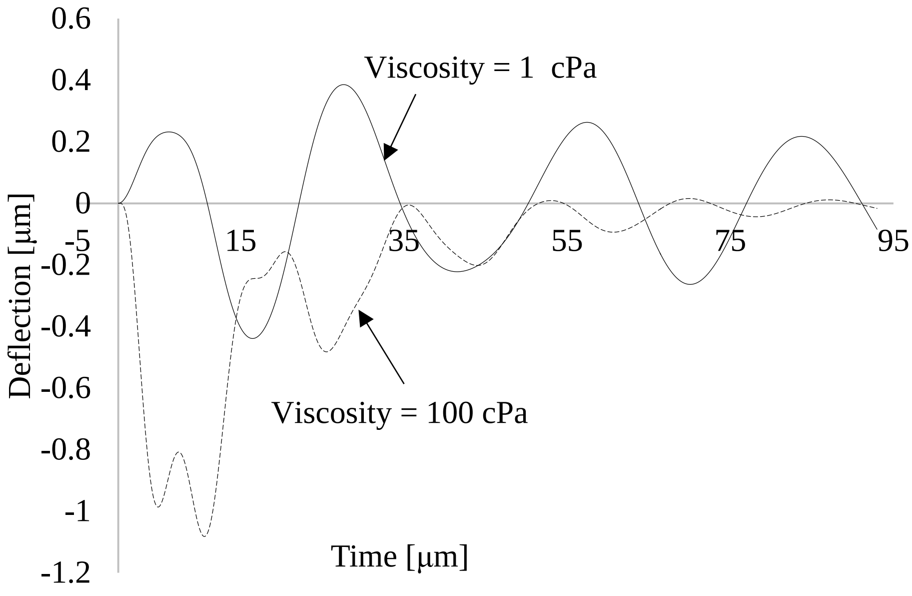

Fig. 5: Response of the droplet under initial deflection of 1 [μm] and two viscosities.

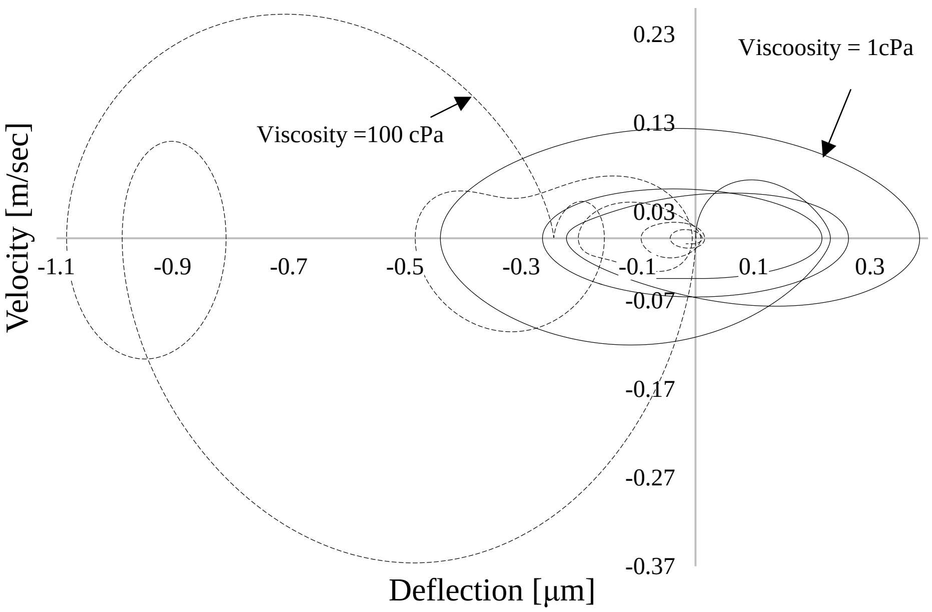

Fig. 6: Phase diagram of the droplet under initial deflection of 1 [μm] and two viscosities.

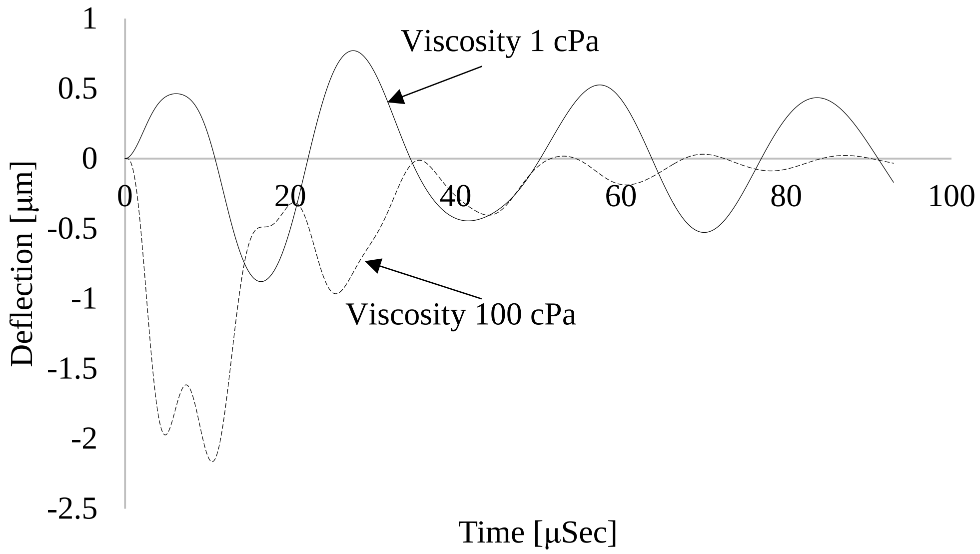

Fig. 7: Time response of the droplet under initial deflection of 2[μm] for two viscosities.

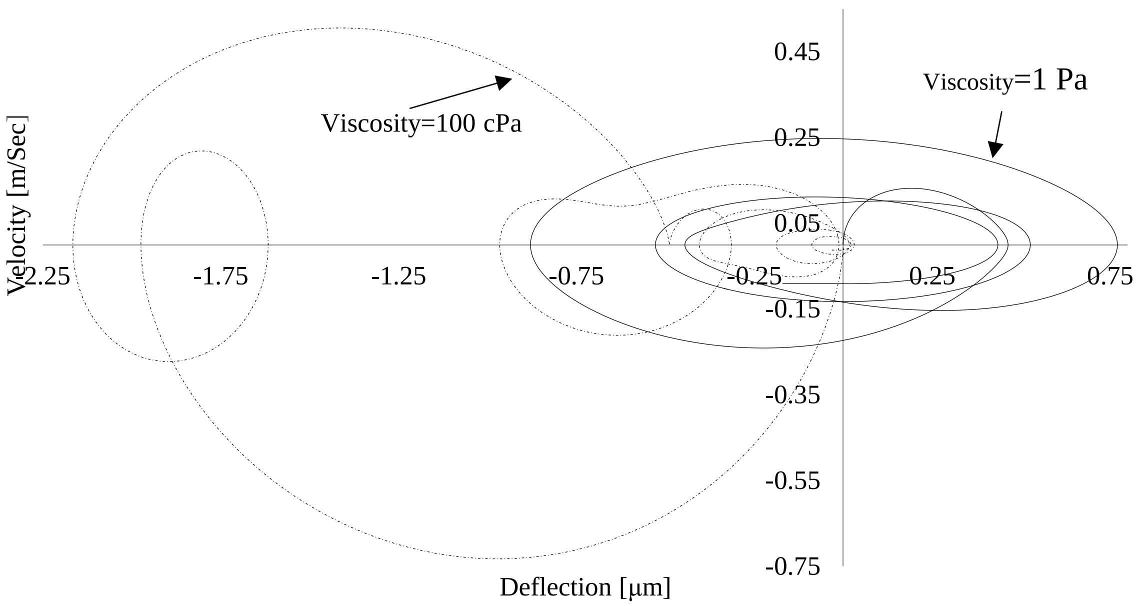

Fig. 8: Phase diagram of the droplet under initial deflection of 2[μm] for two viscosities of the droplet.

5. Conclusion

The present investigation represents a first step towards the development of a novel approach to measure the viscosity of a fluid. A lump mass three degree of freedom model was used in formulation. Specific suitable sizes were assumes as well as a feasible evaporation model of a fluid was considered. The formulation resulted in three coupled non-linear differential equations which were solved using a numerical method – LSODE initial condition based numerical algorithm. The implications of this work could be extended to practical applications.

The methods presently used to measure viscosity require a significantly larger amount of fluid rather than 0.5 pl. In bio-chemistry there are situations in which such a system would be very useful for very rare and expensive reactants. However, the practical approach of this principle is still remote as it raises significant challenges. The first is the detection of the dynamic performance of the c.g. of the droplet. Second major issue consists of matching a reasonable exponential evaporation effect or provide the operation the suitable temperature-RH to make sure that the potential evaporation of the droplet is avoided. Once an experimental setup could be achieved, experiments and validations against standard measurements could be carried out.

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