Proceedings of the 10th International Conference of Control Systems, and Robotics (CDSR'23)
Niagara Falls, Canada – June 01 - 03, 2023
Paper No. 197
DOI: 10.11159/cdsr23.197

Stability Robustness Analysis of Partial Feedback Linearization for a Class of Uncertain Discrete-Time Systems

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Abstract: A class of uncertain nonlinear discrete-time systems is considered for robustness analysis of partial feedback linearization control based on known nonlinearities. It is shown that asymptotic stability of the origin can be maintained in the presence of unknown nonlinear perturbations with growth bounds inversely proportional to the square root of the order of the system. An analytic expression is derived using linear matrix inequality techniques. The theoretical result is verified by extensive MATLAB simulations as to the tightness of the bounds presented.

Keywords: Feedback linearization control, robustness, nonlinear perturbations, linear matrix inequality

1. Introduction

This study is on the robustness of the feedback or global linearization control applied to a class of uncertain nonlinear discrete-time system models. State-space and input-output linearization are now well-established control techniques [1]. They have been used in many applications such as robotics [2] and chemical process control [3] among many. The discrete-time version of feedback linearization was presented in [4] for both input-output and state-space linearization.

This technique has a known robustness issue since knowledge of the nonlinear dynamics is never exact. Therefore, robustification of feedback linearization by other control methods have been proposed in the literature. For example, adaptive control is used in conjunction with feedback linearization in [5], while [6] uses H_∞ control together with input-output linearization for discrete-time systems given by Takagi-Sugeno fuzzy models. In [7], a learning algorithm is used to estimate and cancel nonlinearities. A recent comparison of several robustified feedback controllers can be found in [8].

Robustification effectiveness requires either matching conditions to be known for the uncertain part or extensive data-processing to result in an external representation of the uncertainty to aid in the robust control. In this present paper, we assume neither and instead *analyse the inherent robustness of the discrete-time feedback linearization* scheme based on known nonlinearities as applied to a class of nonlinear discrete-time systems with unmodelled dynamics. Work in this vein can be found, e.g. in [9]-[11] with applications to robotic control.

In the present work, the choice of our model and particular knowledge regarding the nature of the nonlinearities allow for the design of a control resulting in a linear discrete-time system in controllable form with all its eigenvalues at the origin where the uncertainty appears as an additive perturbation with a growth bound. The stability robustness of this feedback linearized system to unknown nonlinear perturbations is analysed. Linear matrix inequality (LMI) techniques [12] are used to derive an expression for the growth bound on the nonlinearities, whose tightness is investigated in extensive simulations.

2. Problem Formulation

The following class of nonlinear discrete-time systems is considered for this study:

$$y_k = g(x_k) + h(x_k)u_k \tag{1}$$

where

$$x_k = \left[y_{k-n}, y_{k-n+1}, \ldots, y_{k-1}\right]^{\mathrm{T}} \tag{2}$$

and u_k, y_k are scalar inputs and outputs. The nonlinear functions, g and h, are partially known. The function g takes the following form

$$g(x_k) = g_1(x_k) + g_2(x_k) \tag{3}$$

where g_1 is the known and g_2 is the unknown part. For h, there are two choices for the unknown part which can be either additive or multiplicative,

$$h(x_k) = h_1(x_k) + h_2(x_k) \quad \text{or}$$
$$h(x_k) = h_1(x_k) h_2(x_k). \tag{4}$$

These nonlinearities satisfy the following bounds, which are assumed to be known but the actual forms of the nonlinearities are not assumed to be known

$$\left| g_i(x_k) \right| \le \beta_i \left\| x_k \right\|$$
$$0 < \gamma_i^{\min} \le \left| h_i(x_k) \right| \le \gamma_i^{\max}. \tag{5}$$

Using the definition of the state vector in (2), equation (1) is written in state-space form as

$$x_{k+1} = Ax_k + B(g(x_k) + h(x_k)u_k) \tag{6a}$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & 0 & ... & ... \\ 0 & 0 & ... & ... & 1 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ ... \\ 0 \\ 1 \end{bmatrix} \tag{6b}$$

Using the feedback linearization control based on the known dynamics:

$$u_k = -\frac{g_1(x_k)}{h_1(x_k)} \tag{7}$$

yields

$$x_{k+1} = Ax_k + Bf_k \tag{8a}$$

with

$$f_k = f_k^a = g_2(x_k) - h_2(x_k)\frac{g_1(x_k)}{h_1(x_k)} \tag{8b}$$

for the additive g_2, and

$$f_k = f_k^m = g_1(x_k)(1 - h_2(x_k)) + g_2(x_k) \tag{8c}$$

for the multiplicative h_2.

Therefore, a unified approach to stability robustness for both types of h_2 is possible with a single upper bound α as follows:

$$\left| f(x_k) \right| \le \alpha \left\| x_k \right\| \tag{9}$$

for x_k in a ball around the origin $B(0,r) \subset \mathbf{R}^n$ where

$$\left| f_k^a \right| \le \left(\beta_2 + \frac{\gamma_2^{\max}\beta_1}{\gamma_1^{\min}} \right) \left\| x_k \right\| = \alpha \left\| x_k \right\| \tag{10a}$$

or

$$\left| f_k^m \right| \le \left(\beta_1 \left(1 + \gamma_2^{\max} \right) + \beta_2 \right) \left\| x_k \right\| = \alpha \left\| x_k \right\| \tag{10b}$$

where α can be considered as the maximum slope of an equivalent nonlinearity composed of known and unknown nonlinear functions given by (5).

This bound can also be expressed in matrix inequality form as

$$\begin{bmatrix} x_k^T & f_k^T \end{bmatrix} \begin{bmatrix} \alpha^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x_k \\ f_k \end{bmatrix} \ge 0. \tag{11}$$

The analysis of the stability is based on the discrete-time algebraic Lyapunov matrix equation

$$P - A^T PA = Q \tag{12}$$

with P > 0 and Q set to the identity matrix, I. The P matrix that solves the Lyapunov equation for a linear, deadbeat control system in canonical form is found as

$$P = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{bmatrix}. \tag{13}$$

Theorem 1: Given the model (1) and (2) of a nonlinear discrete-time system with the bounds on the nonlinearities given in (5) and the partial feedback linearization scheme given in (7), the fixed point 0 is asymptotically stable if the maximum slope of the nonlinearity, α, in (9), (10a) and (10b), satisfies $\alpha < \frac{1}{\sqrt{n}}$.

Proof: Given the system described in (1), using the Lyapunov function candidate

$$V_k = x_k^T P x_k \tag{14}$$

and

$$\Delta V_k = V_{k+1} - V_k < 0 \tag{15}$$

we obtain

$$(Ax_k + Bf_k)^T P(Ax_k + Bf_k) - x_k^T P x_k < 0 \tag{16a}$$

The inequality of 16a can also be expressed as

$$\begin{bmatrix} x_k{}^T & f_k{}^T \end{bmatrix} \begin{bmatrix} P - A^T PA & -A^T PB \\ -B^T PA & -B^T PB \end{bmatrix} \begin{bmatrix} x_k \\ f_k \end{bmatrix} > 0 \tag{16b}$$

The canonical form representation of the system from (6b) and the Lyapunov equation result from (12) and (13) allow the inequality to further be simplified to

$$[x^T{}_k \quad f^T{}_k] \begin{bmatrix} I & 0 \\ 0 & -n \end{bmatrix} \begin{bmatrix} x_k \\ f_k \end{bmatrix} > 0. \tag{17}$$

The S-procedure given in [12] is applied to the inequalities of (11) and (17) in order to combine the system dynamics with the bounds on the nonlinearities. The resulting LMI is

$$\begin{bmatrix} (1 - \tau\alpha^2)I & 0 \\ 0 & \tau - n \end{bmatrix} > 0 \tag{18}$$

where τ is an intermediate variable. LMI (18) yields the conditions $\frac{1}{\alpha^2} > \tau$ and $\tau > n$ which, when combined, results in

$$n < \frac{1}{\alpha^2}. \tag{19}$$

Therefore, the theoretical maximum slope of the nonlinearity is

$$\alpha_{\max} < \frac{1}{\sqrt{n}}. \tag{20}$$

which concludes the proof.

Comment 1. Theorem 1 is presented for nonlinearities with *local* slope bounds given by (9), however, simulation results will show that these results can easily be extended to nonlinearities with *global* bounds.

Comment 2. This result shows that for a system of dimension n, the maximum bound on the nonlinearity for which it is possible to maintain stability for a discrete-time system equals the inverse of the square root of the dimension. Note that for n=1, the theoretical maximum slope of the nonlinearity, α_{max} must be less than 1. As the order of the system increases, the theoretical *maximum slope or the severity* of the nonlinearity that can be accommodated, α_{max}, decreases.

3. Simulations and Results

Case Study 1 This section contains MATLAB simulations that show the results of applying the analysis technique to a nonlinear system whose known part has already been linearized.

Consider a nonlinear system of a form described by the state-space model below.

$$x_{k+1} = Ax_k + Bf_k$$
$$y_k = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} x_k$$
$$f_k = \alpha \|x_k\| \tanh(\frac{\pi}{2}\|x_k\|).$$

The nonlinearity chosen is globally linearly bounded for all its arguments, as shown in Figure 1. The dotted line represents the linear bound on the nonlinearity with a slope α. The nonlinearity, represented by the solid line, has a limit approaching the linear bound as x goes to infinity. This means the system is guaranteed to be *globally asymptotically stable* if $\alpha<\alpha_{max}$.

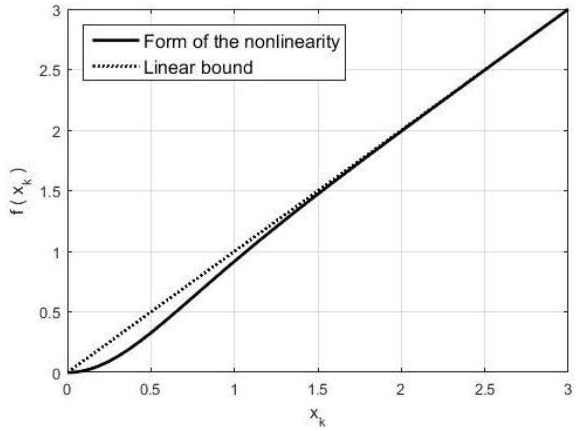

Fig. 1. The form of the nonlinearity and the linear bound

In the following study, the system is simulated with values of α ranging from 0.9 αmax to 1.1 αmax for systems of orders 1, 3, and 5. The value for α_{max} corresponds to each value for n, as shown in Table 1.

Table 1: A table of system orders and the corresponding theoretical α_{max}.

n	1	3	5
α_{max}	1	0.577	0.447

Fig. 2 is a plot of the output, yk as a function of k for systems with a value of α just below and just above that of αmax. This figure confirms that for nonlinearities that are bounded with α<αmax, the system is asymptotically stable. For nonlinearities that exceed this bound, stability is not guaranteed. Therefore, for this specific nonlinearity, our theoretical sufficient result is almost necessary.

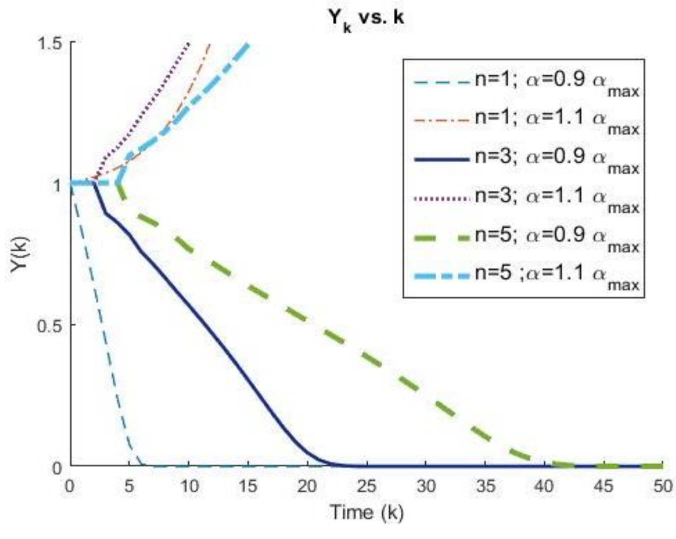

Fig. 2. The output over time for systems of order 1, 3, and 5. The bound is below and above the theoretical bound

Case Study 2:

Example 1: Consider a 2nd order nonlinear system described by the state-space model below.

$$x_{k+1}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}x_k+\begin{bmatrix}0\\ 1\end{bmatrix}\alpha\left\|x_k\right\|^3$$

$$y_k=\begin{bmatrix}1 & 0\end{bmatrix}x_k\,.$$

Since this is a 2nd order system, the maximum guaranteed bound on the nonlinearity is calculated to be

$$\alpha_{\max} = \frac{1}{\sqrt{2}} \cong 0.7071.$$

Note that the bound on this nonlinearity is local. The system is simulated with values of α ranging from 0 to 2 $\alpha_{\max}$. The plots show the nonlinearity, F_k, versus the norm of x_k and the output, y_k versus k for three different sets of initial conditions. The straight red line in the nonlinear plot represents the maximum linear bound as a function of the norm of x_k while the blue circle represents the initial norm of x_k. The black dotted lines represent the track of the nonlinearities for each value of α

For initial conditions, $x_0 = 0.5 * \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$ the simulation results are shown in Fig. 3. For initial conditions, $x_0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$, the simulation results are shown in Fig 4. And finally, for initial conditions $x_0 = 1.5 * \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$, the simulation results are shown in Fig 5.

This nonlinearity grows as a function of x in such a way that it cannot be linearly bounded. Therefore, the result cannot be globally asymptotically stable. However, the cubic nonlinearity does have a region of convergence for $\|x_k\| \leq 1$. From the simulations, it is seen that when $\|x_o\| \leq 1$, the system is locally asymptotically stable when $\alpha < \alpha_{\max}$.

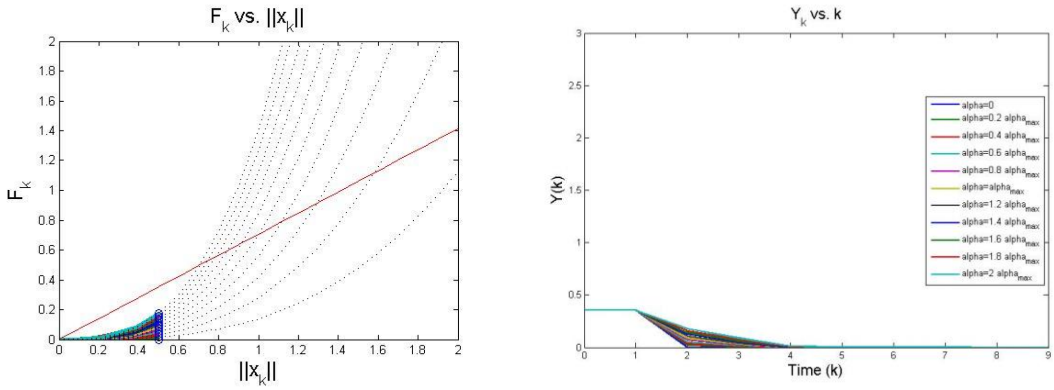

Fig. 3. Nonlinearity and output for the second order system when initial conditions are less than the bound.

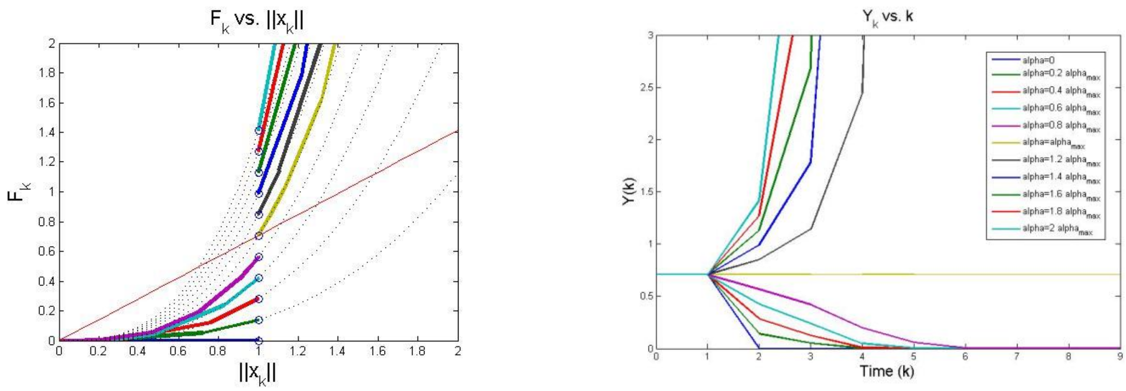

Fig. 4. Nonlinearity and output for the second order system when initial conditions are equal to the bound.

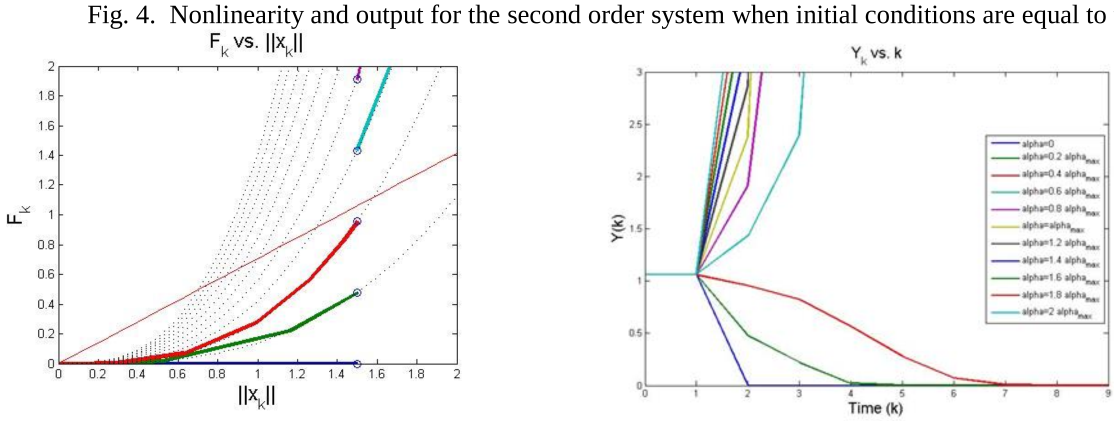

Fig. 5. Nonlinearity and output for the second order system when initial conditions are greater than the bound.

Example 2: Consider a nonlinear system described by the state-space model,

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \alpha \|x_k\| \tanh(\frac{\pi}{2}\|x_k\|)$$

$$y_k = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_k .$$

Since this is a 3rd order system, the maximum guaranteed bound on the nonlinearity is calculated to be

$$\alpha_{\max} = \frac{1}{\sqrt{3}} \cong 0.5774.$$

For the following figures, the system is simulated with values of α ranging from 0 to 2 $\alpha_{\max}$. As before these plots show the nonlinearity versus the norm of x_k and the output, y_k versus k. The straight red line in the plot represents the maximum linear bound as a function of the norm of x_k while the blue circle represents the initial norm of x_k. The black dotted lines represent the track of the nonlinearities for each value of α. The simulation results for three different sets of initial conditions below, equal to and greater than the bound for the 3rd order system are shown in

Fig. 6 where $x_0 = 0.5*\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T$, Fig. 7 for $x_0 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T$ and Fig. 8 when $x_0 = 1.5*\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T$.

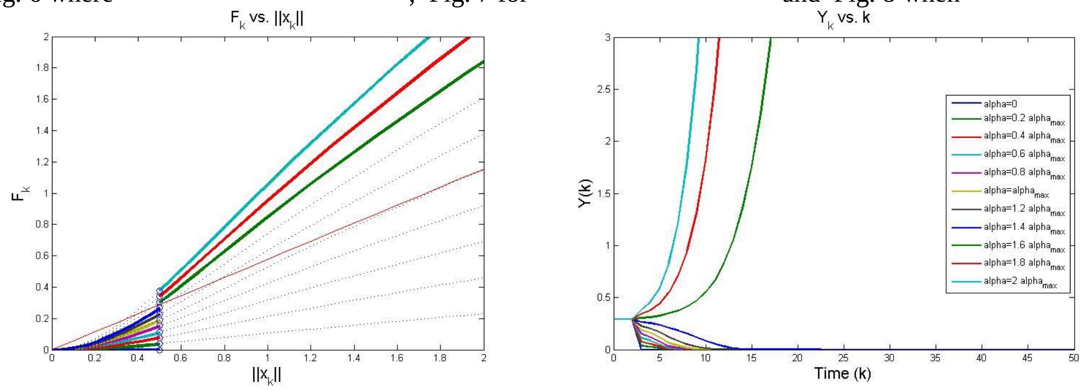

Fig. 6. Nonlinearity and output for the third order system when initial conditions are less than the bound.

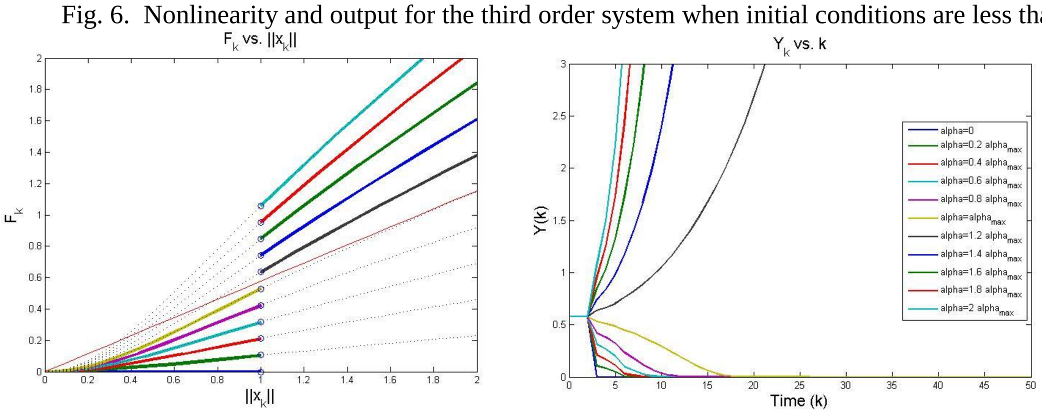

Fig. 7. Nonlinearity and output for the third order system when initial conditions equal the bound.

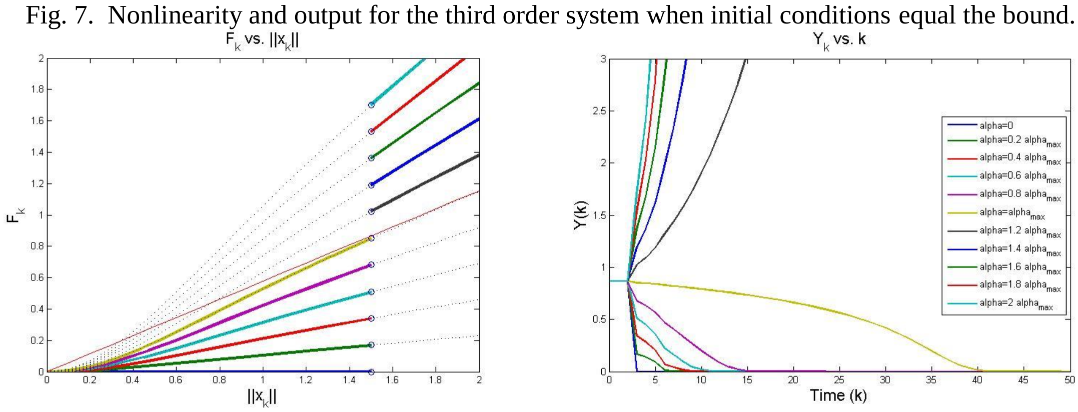

Fig. 8. Nonlinearity and output for the third order system when initial conditions are greater than the bound.

The results for the 3rd order system clearly show that when the norm of x_k stays below the red line with slope α_{max}, the system maintains asymptotic stability. This holds true for all cases where $\alpha<\alpha_{max}$. For some small initial conditions, some values of $\alpha>\alpha_{max}$ also produce outputs which are asymptotically stable, but it cannot be guaranteed for all cases. The nonlinearity increases in such a way that it can be globally bounded with a slope bound. Therefore, in this case, for $\alpha<\alpha_{max}$, the system is globally asymptotically stable.

Example 3: Consider a 4th order nonlinear system described by the state-space model below.

$$x_{k+1}=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}x_k+\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\alpha\left\|x_k\right\|sin(\frac{\pi}{2}\left\|x_k\right\|)$$

$$y_k=\begin{bmatrix}1&0&0&0\end{bmatrix}x_k.$$

Since this is a 4th order system, the maximum guaranteed bound on the nonlinearity is calculated to be

$$\alpha_{\max}=\frac{1}{\sqrt{4}}=0.5.$$

As for the previous examples, the system is simulated with values of α ranging from 0 to 2 α_{max} where the plots demonstrate nonlinearity versus the norm of x_k and the output, y_k versus k. The straight red line in the nonlinear plot represents the maximum linear bound as a function of the norm of x_k while the blue circle represents the initial norm of x_k. The black dotted lines represent the track of the nonlinearities for each value of α. Fig. 9 corresponds to results obtained when the initial conditions are $x_0=0.5*\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{bmatrix}^T$. When the initial conditions are $x_0=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{bmatrix}^T$, the simulation results appear as shown in Fig. 10 where Fig 11 shows the results for the situation when $x_0=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{bmatrix}^T$.

From these results, it is again seen that for $\alpha<\alpha_{max}$, the output is asymptotically stable in all cases. For a small initial condition, some values of $\alpha>\alpha_{max}$ also produce outputs which are asymptotically stable. However, it cannot be guaranteed for all cases. The nonlinearity increases as a function of x in such a way that it can be globally bounded. Therefore, in this case, for $\alpha<\alpha_{max}$, the system is globally asymptotically stable.

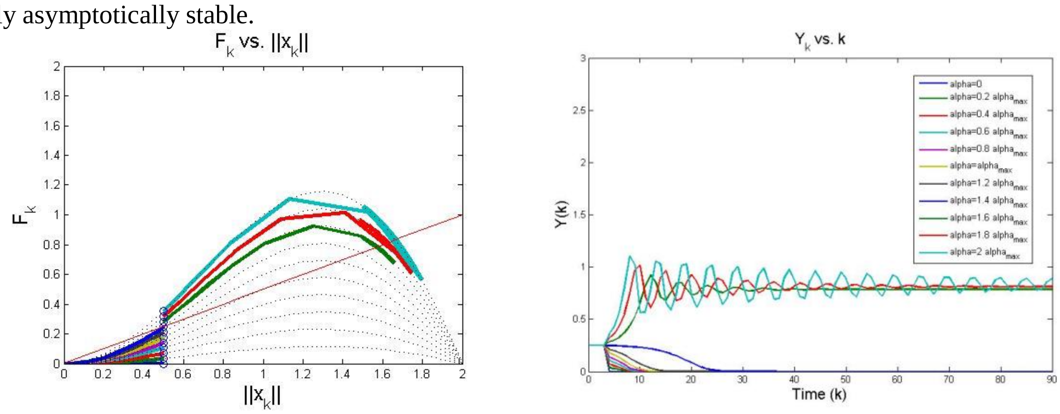

Fig. 9. Nonlinearity and output for the fourth order system when initial conditions are less than the bound.

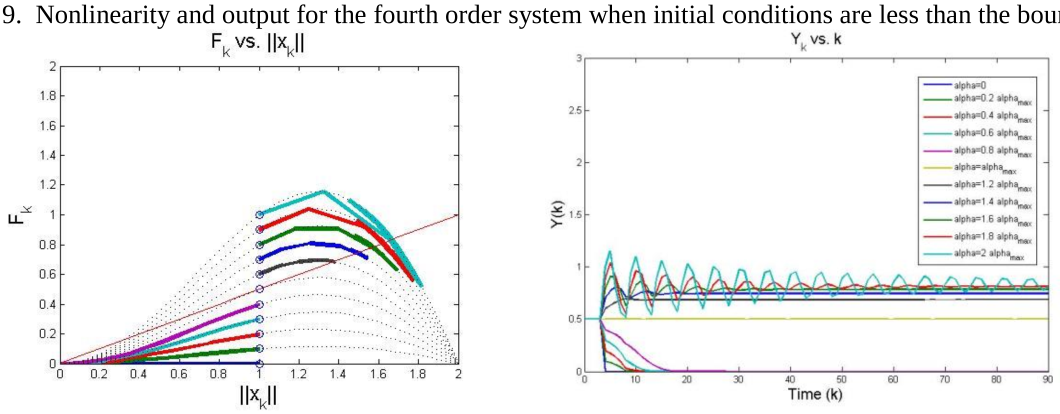

Fig. 10. Nonlinearity and output for the fourth order system when initial conditions are equal to the bound.

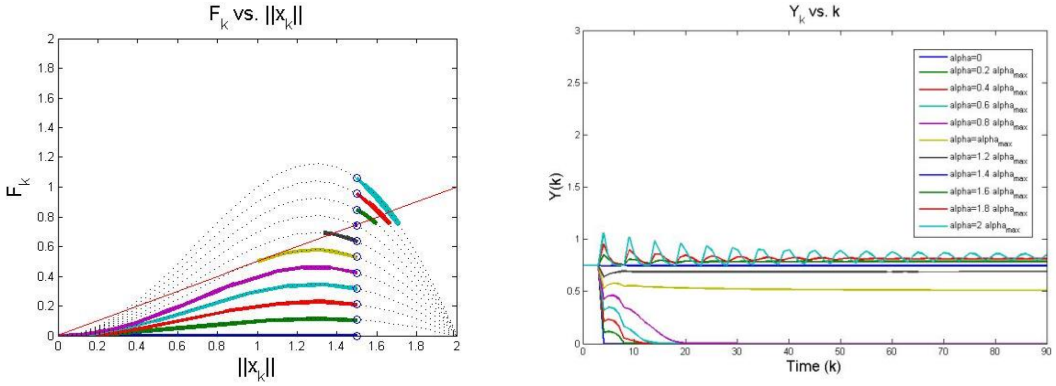

Fig. 11. Nonlinearity and output for the fourth order system when initial conditions are greater than the bound.

4. Conclusions

In this paper, the robustness of the feedback linearization control scheme was considered for discrete-time systems where the nonlinearities are partially unknown with known bounds. An analysis of the partially feedback linearized system revealed an interesting relationship between the maximum guaranteed growth bound on the uncertainty and the order of the system. This relationship was proven analytically to be a sufficient condition for guaranteeing asymptotic stability and tested in simulation on nonlinear systems of varying orders and with various nonlinearities to demonstrate that it is also close to being a necessary condition for maintaining stability. The future work will include expanding these results to D-stability robustness analysis of the linearized part in a way similar to [13] rather than the robustness of the partially stabilized system with all eigenvalues at the origin. This will necessitate the circular region to be defined based on a modified Lyapunov inequality.

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