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Geometric Control of a Quadrotor with Attitude Control on Unit Circles

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Abstract **–** This paper presents a geometric controller, consists of an altitude and separated attitude control components, for trajectory tracking purpose suitable for quadrotors. To facilitate easier gain tuning, the proposed controller is developed, so that the roll, pitch and yaw controllers are separated from each other. Meanwhile, the inner attitude controller is developed on SO(3), which prevents singularities or ambiguities arising from using minimal representations.

Keywords**:** Geometric Control, Unmanned Aerial Vehicles (UAVs), Trajectory Tracking, Attitude Control

1. Introduction

Quadrotor unmanned aerial vehicles (UAVs), known for their mechanically simple design yet capable of vertical take-off and landing (VTOL), have gained popularity across various applications such as surveillance and aerial transportation [1], [2], [3]. However, due to mechanical constraints, thrust can only be generated along the body z axes. Attitude controllers developed based on Euler angles are prone to singularities while those developed based on quaternions exhibit ambiguities due to the double-covering behaviour. In contrast, geometric control eliminates these disadvantages by developing the controller directly on SE(3).

In regard to control inputs, although quadrotors are generally considered to be able to generate moment arbitrarily, the resultant moment about the body z axis is typically orders of magnitude lower than the other axes, because it is generated by reaction torques rather than differences in thrust over moment arm. As a result, achieving moderate yawing moment requires significant and rapid changes in rotor speeds. This not only risks impacting rolling and pitching dynamics as rotor responses will not be perfect but may also lead to saturation issues [2], [3]. To address this problem, there is a desire to separate yaw from roll and pitch control as shown in [2], [3]. Building upon this work, this paper further separates the roll and pitch control, where the separated roll, pitch and yaw controllers are all developed on unit circles, facilitating easier gain tuning without compromising the performance. Additionally, the controller design proposed in this paper relaxes the assumption of identical moments of inertia about the x and y axes, providing a versatile framework applicable not only to standard quadrotors, but also to non-square quadrotors, other multi-rotor configurations, and UAV systems with payloads or movable components.

2. Problem Formulation

2.1. Dynamic Model of a Quadrotor UAV

In this paper, the following dynamic model for a quadrotor is considered. The inertial frame (e_x, e_y, e_z) and the vehicle body frame (b_x, b_y, b_z) are defined following a North-East-Down convention. Assuming thrust, weight and motor torque are dominant compared to all other external forces and moments, such as drag and wall effect, the kinematics and dynamics of the quadrotor position and attitude can be modelled as follows,

$$\dot{p} = v, \tag{1}$$
$$m\dot{v} = R[0 \quad 0 \quad -T]^T + [0 \quad 0 \quad mg]^T, \tag{2}$$
$$\dot{R} = R\hat{\omega}, \tag{3}$$
$$J\dot{\omega} + \omega \times (J\omega) = M, \tag{4}$$
$$R = [b_x \quad b_y \quad b_z], \tag{5}$$

$$\begin{bmatrix} T \\ M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -d & 0 & d \\ d & 0 & -d & 0 \\ -C_{QT} & C_{QT} & -C_{QT} & C_{QT} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \mathbb{T} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}, \tag{6}$$

where hat map $\widehat{\ }: \mathbb{R}^3 \to SO(3)$ is defined as $\hat{x}y = x \times y$, and $p \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$are linear position and velocity vectors in the inertial frame, $R \in SO(3)$ is rotational matrix from the body frame to the inertial frame, $\omega \in \mathbb{R}^3$ is angular velocity vector in the body frame, $mg \in \mathbb{R}$ and $J \in \mathbb{R}^{3\times 3}$ are weight and moment of inertia matrix of the quadrotor UAV, $T \in \mathbb{R}$ and $M = [M_x \quad M_y \quad M_z]^T \in \mathbb{R}^3$ are sum of thrust from all four rotors and vector of sum of external moments in the body frame, $d \in \mathbb{R}$ is distance from the quadrotor UAV center of mass to the center of each rotor, $T_i \in \mathbb{R}$ and $C_{QT} \in \mathbb{R}$ are thrust from the i-th rotor and ratio of rotor torque to thrust, $\mathbb{T} \in \mathbb{R}^{4\times 4}$ maps T_i to T and M. Subscripts x, y and z are to show element around first, second and third basis and subscripts d and c are to indicate desired and computed values.

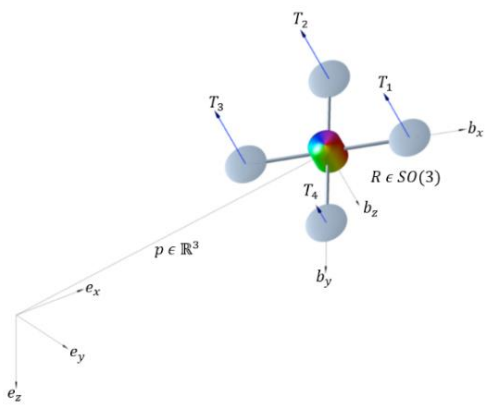

Fig. 1: Illustration of the Quadrotor Model [1]

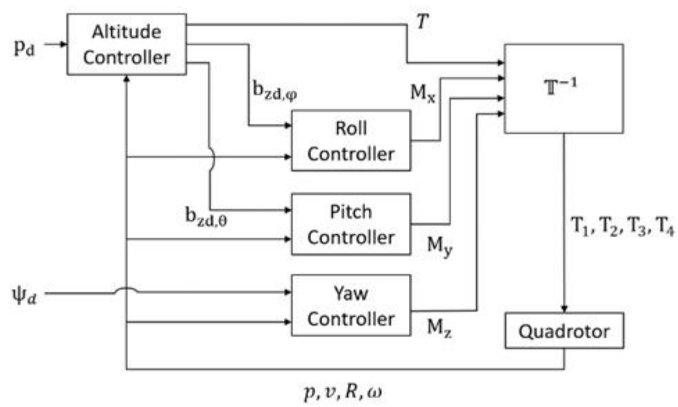

Fig. 2: Block Diagram of the Proposed Controller

With a diagonal inertia matrix, $J = \mathrm{diag}(J_{xx}, J_{yy}, J_{zz})$, note there is no structural requirement for $J_{xx} = J_{yy}$ as was in [2], [3], Eq. (4) can be re-written as

$$J_{xx}\dot{\omega}_x = M_x - (J_{zz} - J_{yy})\omega_y\omega_z, \tag{7}$$
$$J_{yy}\dot{\omega}_y = M_y - (J_{xx} - J_{zz})\omega_x\omega_z, \tag{8}$$
$$J_{zz}\dot{\omega}_z = M_z - (J_{yy} - J_{xx})\omega_x\omega_y. \tag{9}$$

2.2. Control Objective

The objective of the proposed controller is to determine the thrust of each rotor (T_1, T_2, T_3, T_4), under certain assumptions described in Section 3, such that the actual trajectory, $p(t)$, and the body x axis, $b_x(t)$, follows an arbitrary desired trajectory, $p_d(t)$, and a converted body x axis, $b_{xc}(t)$, asymptotically. $b_{xc}(t)$, is obtained by converting an arbitrary desired yaw angle, $\psi_d(t)$, into a vector for tracking purpose as explained in Section 3.5.

With Eq. (6), there is a one-to-one transformation between the thrust of each rotor (T_1, T_2, T_3, T_4) and the sum of thrust and moments (T, M_x, M_y, M_z) as long as $d \neq 0$ and $C_{QT} \neq 0$, which should always be true. Therefore, the control objective is equivalent to the determination of the desired sum of thrust and external moment (T, M_x, M_y, M_z).

3. Controller Design

The proposed controller follows a similar approach in [1], [2], [3], but a separated roll, a separated pitch and a separated yaw controller are used for the attitude control.

3.1. Position Controller

In [1] , the desired external force vector is defined as $A \in \mathbb{R}^3$,

$$A = -k_p e_p - k_v e_v - mg e_z + m a_d, \tag{10}$$

where k_p and k_v are positive control gains, and $e_p = p - p_d$, $e_v = v - v_d$, $v_d = \dot{p}_d$ and $a_d = \dot{v}_d$. Due to mechanical constrains, the sum of thrust of a quadrotor is always pointing along the negative body z axis, $-b_z$, and has a magnitude of $T \in \mathbb{R}$. Therefore, A is tracked as long as $-b_z \to -b_{zd} = \frac{A}{\|A\|}$ and $T \to -A \cdot b_z$. The former part can be defined as the control objective of the attitude controller while the latter part can be defined as the control objective of the altitude controller.

3.2. Altitude Controller

The control objective of the altitude controller is to make T tracks $-A \cdot b_z$. Since T is a control input, we can set

$$T = -A \cdot b_z. \tag{11}$$

3.3. Separated Roll Controller

The control objective of the attitude controller is to make $-b_z$ tracks $-b_{zd} = \frac{A}{\|A\|}$. From this, the roll error function and its derivative can then be defined similar to [2], [3], [4] as follows,

$$\Psi_\varphi = \frac{1}{2}\left\|b_z - b_{zd,\varphi}\right\|^2 = 1 - b_z \cdot b_{zd,\varphi}, \tag{12}$$

$$\dot{\Psi}_\varphi = e_\varphi e_{\dot{\varphi}}, \tag{13}$$

where $b_{zd,\varphi} = \frac{b_x \times (b_x \times A)}{\|b_x \times (b_x \times A)\|}$, $e_\varphi = b_{zd,\varphi} \cdot b_y$, $e_{\dot{\varphi}} = \omega_x - \omega_{zd,\varphi,x}$, $\omega_{zd,\varphi,x} = \omega_{zd,\varphi} \cdot b_x$ and $\omega_{zd,\varphi} = b_{zd,\varphi} \times \dot{b}_{zd,\varphi}$. By considering the following Lyapunov candidate,

$$V_\varphi = \frac{1}{2} J_{xx} {e_{\dot{\varphi}}}^2 + c_\varphi J_{xx} e_\varphi e_{\dot{\varphi}} + k_\varphi \Psi_\varphi, \tag{14}$$

where k_φ and $k_{\dot{\varphi}}$ are positive control gains and c_φ is a non-negative constant, the equilibrium $(e_\varphi, e_{\dot{\varphi}}) = (0,0)$ can be proven to be (asymptotically) stable by defining $\dot{e}_{\dot{\varphi}}$ as follows,

$$J_{xx} \dot{e}_{\dot{\varphi}} = -k_\varphi e_\varphi - k_{\dot{\varphi}} e_{\dot{\varphi}}. \tag{15}$$

Meanwhile, by differentiating $e_{\dot{\varphi}} = \omega_x - \omega_{zd,\varphi,x}$ and multiply it by J_{xx}, we can obtain

$$J_{xx} \dot{e}_{\dot{\varphi}} = J_{xx} \dot{\omega}_x - J_{xx} \dot{\omega}_{zd,\varphi,x}. \tag{16}$$

With Eq. (7) and Eq. (15), and rearranging them, the desired M_x can then be defined as follows,

$$M_x = -k_\varphi e_\varphi - k_{\dot{\varphi}} e_{\dot{\varphi}} + (J_{zz} - J_{yy}) \omega_y \omega_z + J_{xx} \dot{\omega}_{zd,\varphi,x} \tag{17}$$

Please refer to Section 6.1 for the proof.

3.4. Separated Pitch Controller

Similar to Section 3.3, the pitch error function can then be defined as follows,

$$\Psi_\theta = \frac{1}{2}\left\|b_z - b_{zd,\theta}\right\|^2 = 1 - b_z \cdot b_{zd,\theta}, \tag{18}$$

$$\dot{\Psi}_\theta = e_\theta e_{\dot{\theta}}, \tag{19}$$

where $b_{zd,\theta} = \frac{b_y \times (b_y \times A)}{\|b_y \times (b_y \times A)\|}$, $e_\theta = -b_{zd,\theta} \cdot b_x$, $e_{\dot{\theta}} = \omega_y - \omega_{zd,\theta,y}$, $\omega_{zd,\theta,y} = \omega_{zd,\theta} \cdot b_y$ and $\omega_{zd,\theta} = b_{zd,\theta} \times \dot{b}_{zd,\theta}$. By considering the following Lyapunov candidate,

$$V_\theta = \frac{1}{2} J_{yy} {e_{\dot{\theta}}}^2 + c_\theta J_{yy} e_\theta e_{\dot{\theta}} + k_\theta \Psi_\theta, \tag{20}$$

where k_θ and $k_{\dot{\theta}}$ are positive control gains and c_θ is a non-negative constant, the equilibrium $(e_\theta, e_{\dot{\theta}}) = (0,0)$ can be proven to be (asymptotically) stable by defining $\dot{e}_{\dot{\theta}}$ as follows,

$$J_{yy}\dot{e}_{\dot{\theta}} = -k_\theta e_\theta - k_{\dot{\theta}} e_{\dot{\theta}}. \tag{21}$$

Meanwhile, by differentiating $e_{\dot{\theta}} = \omega_y - \omega_{zd,\theta,y}$ and multiply it by J_{yy}, we can obtain

$$J_{yy}\dot{e}_{\dot{\theta}} = J_{yy}\dot{\omega}_y - J_{yy}\dot{\omega}_{zd,\theta,y}. \tag{22}$$

With Eq. (8) and Eq. (21) and rearranging them, the desired M_y can then be defined as follows,

$$M_y = -k_\theta e_\theta - k_{\dot{\theta}} e_{\dot{\theta}} + (J_{xx} - J_{zz})\omega_x \omega_z + J_{yy}\dot{\omega}_{zd,\theta,y}. \tag{23}$$

Please refer to Section 6.1 for the proof.

3.5. Separated Yaw Controller

Since position control of a quadrotor can be accomplished with (T, M_x, M_y) obtained from the altitude, separated roll and separated pitch controllers described in the Sections 3.2, 3.3 and 3.4. The last control input, M_z, can be utilized for yaw control to track an arbitrary desired yaw angle, ψ_d or, equivalently, an arbitrary desired body x axis,

$$b_{xd} = [\cos(\psi_d) \quad \sin(\psi_d) \quad 0]^T. \tag{24}$$

Since b_{zd} is defined in the position control, b_{xd} cannot always be tracked. Instead, M_z can be utilized to track a converted body x axis, b_{xc}, derived from b_{xd}. The yaw error function can then be defined accordingly as in [2], [3], [4].

$$\Psi_\psi = \frac{1}{2}\|b_x - b_{xc}\|^2 = 1 - b_x \cdot b_{xc}, \tag{25}$$

$$\dot{\Psi}_\psi = e_\psi e_{\dot{\psi}}, \tag{26}$$

where $b_{xc} = -\frac{b_z \times (b_z \times b_{xd})}{\|b_z \times (b_z \times b_{xd})\|}$, $e_\psi = -b_{xc} \cdot b_y$, $e_{\dot{\psi}} = \omega_z - \omega_{xc,z}$, $\omega_{xc,z} = \omega_{xc} \cdot b_z$ and $\omega_{xc} = b_{xc} \times \dot{b}_{xc}$. By considering the following Lyapunov candidate,

$$V_{\psi} = \frac{1}{2} J_{zz} e_{\dot{\psi}}^2 + c_{\psi} J_{zz} e_{\dot{\psi}} e_{\psi} + k_{\psi} \Psi_{\psi}, \tag{27}$$

where k_{ψ} and $k_{\dot{\psi}}$ are positive control gains and c_{ψ} is a non-negative constant, the equilibrium $(e_{\psi}, e_{\dot{\psi}}) = (0,0)$ can be proven to be (asymptotically) stable by defining $\dot{e}_{\dot{\psi}}$ as follows,

$$J_{zz} \dot{e}_{\dot{\psi}} = -k_{\psi} e_{\psi} - k_{\dot{\psi}} e_{\dot{\psi}}. \tag{28}$$

Meanwhile, by differentiating $e_{\dot{\psi}} = \omega_z - \omega_{xc,z}$ and multiply it by J_{zz}, we can obtain

$$J_{zz} \dot{e}_{\dot{\psi}} = J_{zz} \dot{\omega}_z - J_{zz} \dot{\omega}_{xc,z}. \tag{29}$$

With Eq. (9) and Eq. (28) and rearranging them, the desired M_z can then be defined as follows,

$$M_z = -k_{\psi} e_{\psi} - k_{\dot{\psi}} e_{\dot{\psi}} + (J_{yy} - J_{xx}) \omega_x \omega_y + J_{zz} \dot{\omega}_{xc,z}. \tag{30}$$

Please refer to Section 6.1 for the proof.

4. Numerical Simulation

Physical parameters are chosen to match the quadrotor UAV described in [1], where $m = 4.34\ \mathrm{kg}$, $J = \mathrm{diag}(0.0820, 0.0845, 0.1377)\ \mathrm{kg \cdot m^2}$, $d = 0.315\ \mathrm{m}$ and $C_{QT} = 8.004 \times 10^{-3}\ \mathrm{m}$.

Considering a typical propeller blade is usually designed for providing positive thrust and depending on the motor used, there are limits on the minimum and maximum thrust each rotor can provide. With these two mechanical constraints, the minimum and maximum thrust output for each rotor in this numerical simulation are set to $T_{min} = 0$ N and $T_{max} = 25$ N, respectively, which the maximum total thrust output is 100 N and is slightly above twice the weight of the quadrotor UAV. With the limits on the thrust output, a quadrotor UAV could be unstable if the control gains were set too high. This is because the thrust of each rotor (T_1, T_2, T_3, T_4) will become saturated at the minimum and maximum thrust output and cannot track the commanded thrusts.

To illustrate the legitimacy of the separated attitude controller under the limits on the thrust output, all control gains in this numerical simulation are arbitrarily set to 1, i.e. $k_p = k_v = k_R = k_\Omega = k_\varphi = k_{\dot{\varphi}} = k_\theta = k_{\dot{\theta}} = k_\psi = k_{\dot{\psi}} = 1$. The desired trajectory is chosen as a figure eight shape, where the position and yaw angle are defined as follows, $p_d(t) = [\sin(t) \quad \sin(2t) \quad -1 + 0.2\cos(2t)]^T$ m and $\psi_d(t) = \frac{\pi}{5} t$ rad. The initial conditions are set as follows, $p(0) = [0 \quad 0 \quad 0]^T$ m, $v(0) = [0 \quad 0 \quad 0]^T\ \mathrm{m \cdot s^{-1}}$, $R(0) = I_{3\times3}$ and $\omega(0) = [0 \quad 0 \quad 0]^T\ \mathrm{rad \cdot s^{-1}}$.

The numerical simulation results are shown in Fig. 3 to Fig. 6, where "Classical" denotes the position controller proposed in [1] and "Separated" denotes the controller proposed in this paper. As shown in Fig. 3 and Fig. 4, both controllers approach the desired trajectory even under the thrust output limits. Meanwhile, this results also show that the separated attitude controller does not sacrifice in performance compared to the previous work [1]. Fig. 5 and Fig. 6 illustrate the control effort and the separated attitude controller has the same level of control effort compared to the previous work.

5. Conclusion

This paper presents a geometric control system for a quadrotor UAV, which the attitude control is accomplished by three separate controllers. This was achieved by designing each of the roll, pitch and yaw controllers on a unit circle. The numerical simulation result shows that separating the attitude controller does not compromise the performance. Furthermore, this separation allows the deterioration of the rolling and pitching dynamics induced by the yawing moment to be handled explicitly. Meanwhile, this configuration facilitates easier gain tuning and can be adopted for other attitude control purposes.

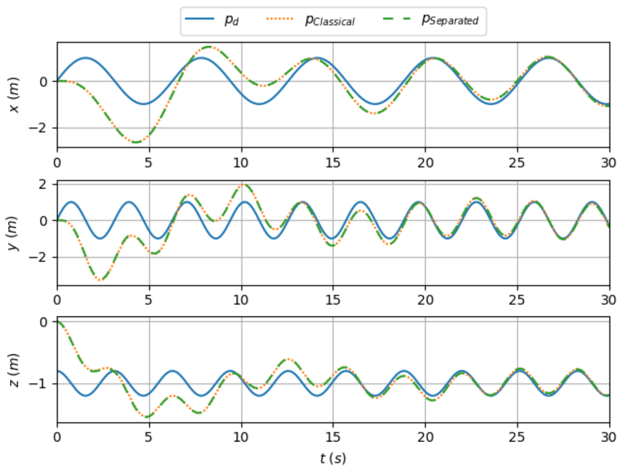

Fig. 3: Position vs Time

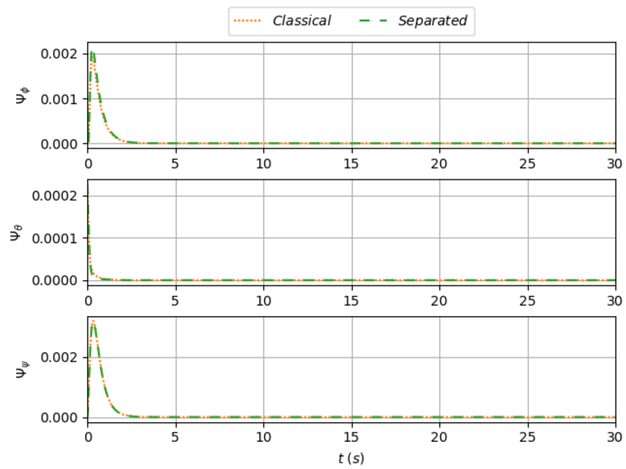

Fig. 4: Attitude Errors vs Time

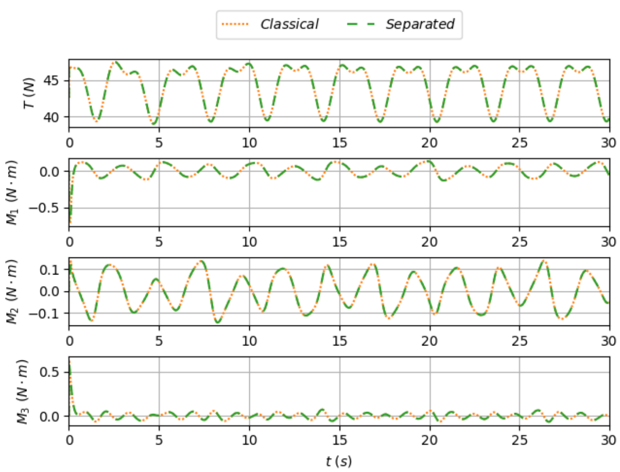

Fig. 5: Control Inputs vs Time

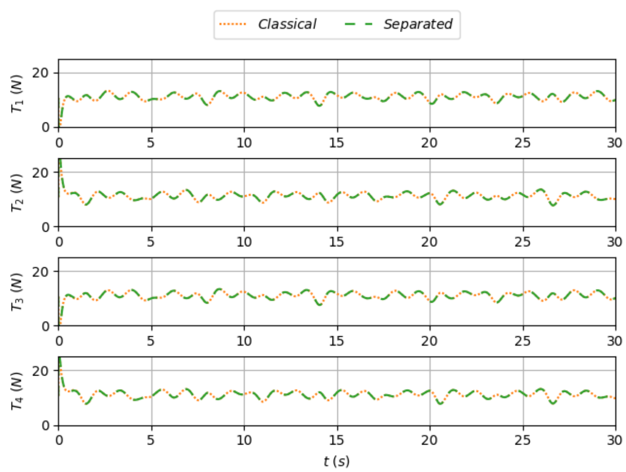

Fig. 6: Thrusts vs Time

6. Appendix: Proofs

Due to the page limit, the proof of asymptotic stability and instability of the other equilibrium for the separated roll, pitch and yaw controller are not included here, but can be shown in a similar fashion as in [2], [3], [4].

6.1. Proof of Stability for the Separated Roll, Separated Pitch, and Separated Yaw Controllers

As per Proposition 2.1 Property (iv) in [4],

$$\Psi_\varphi \geq \frac{1}{2}\left\|b_{zd,\varphi} \times b_z\right\|^2 = \frac{1}{2} e_\varphi{}^2, \tag{31}$$

With Eq. (14) and set $c_\varphi = 0$, V_φ is positive definite. By differentiating Eq. (14) and substituting Eq. (15),

$$\dot{V}_\varphi = -k_{\dot{\varphi}} e_{\dot{\varphi}}{}^2. \tag{32}$$

Therefore, $\dot{V}_\varphi$ is negative semi-definite, and the separated roll controller is stable. The stability of the separated pitch controller and separated yaw controller can be shown in the same fashion.

6.2. Proof of Asymptotic Stability for the Position Controller

As in [1], consider the following Lyapunov candidate for position control,

$$V_p = \frac{k_p}{2} {e_p}^T e_p + c_p m {e_p}^T e_v + \frac{m}{2} {e_v}^T e_v. \tag{33}$$

By taking derivative of the Lyapunov candidate,

$$\dot{V}_p = k_p {e_p}^T e_v + c_p m {e_v}^T e_v + \left(c_p {e_p}^T + {e_v}^T\right)\left(-k_p e_p - k_v e_v\right) + \left(c_p {e_p}^T + {e_v}^T\right)(-T b_z - A), \tag{34}$$

$$\dot{V}_p \leq -X_p^T Q_P X_p + m(g + \|a_d\|)\left(c_p\|e_p\| + \|e_v\|\right)\|A\|\|e_{\varphi\theta}\|, \tag{35}$$

$${e_{\varphi\theta}}^2 = \frac{{A_x}^2 + {A_y}^2}{{A_x}^2 + {A_y}^2 + {A_z}^2} = \frac{\|A \times b_z\|^2}{\|A\|^2} \leq 1, \tag{36}$$

$$X_p = \left[\|e_p\| \quad \|e_v\|\right]^T, \tag{37}$$

$$Q_p = \begin{bmatrix} c_p k_p\left(1 - \|e_{\varphi\theta}\|\right) & \frac{-c_p k_v(1 - \|e_{\varphi\theta}\|) - k_p\|e_{\varphi\theta}\|}{2} \\ \frac{-c_p k_v(1 - \|e_{\varphi\theta}\|) - k_p\|e_{\varphi\theta}\|}{2} & k_v\left(1 - \|e_{\varphi\theta}\|\right) - m c_p \end{bmatrix}. \tag{38}$$

Consider $B_{\varphi\theta} < 1$ to be the upper bound of $\|e_{\varphi\theta}\|$ for Q_p to be positive definite,

$$Q_p\left(\|e_{\varphi\theta}\| = B_{\varphi\theta}\right) = \begin{bmatrix} c_p k_p\left(1 - B_{\varphi\theta}\right) & \frac{-c_p k_v(1 - B_{\varphi\theta}) - k_p B_{\varphi\theta}}{2} \\ \frac{-c_p k_v(1 - B_{\varphi\theta}) - k_p B_{\varphi\theta}}{2} & k_v\left(1 - B_{\varphi\theta}\right) - m c_p \end{bmatrix}. \tag{39}$$

Therefore, $Q_p\left(\|e_{\varphi\theta}\| = B_{\varphi\theta}\right)$ is positive definite if $c_p < \frac{k_v(1 - B_{\varphi\theta})}{m}$ and $4 c_p k_p\left(1 - B_{\varphi\theta}\right)\left(k_v\left(1 - B_{\varphi\theta}\right) - m c_p\right) > \left(-c_p k_v\left(1 - B_{\varphi\theta}\right) - k_p B_{\varphi\theta}\right)^2$. As $B_{\varphi\theta} \to 0$, the two conditions become $c_p < \frac{k_v}{m}$ and $c_p < \frac{4 k_p k_v}{{k_v}^2 + 4 m k_p}$. Since c_p can be chosen to be arbitrarily small, there exists a region where Q_p is positive definite and

$$-\lambda_{max}(Q_P)\|X_p\|^2 \leq -X_p^T Q_P X_p \leq -\lambda_{min}(Q_P)\|X_p\|^2 \tag{40}$$

Meanwhile, from roll and pitch error functions,

$${e_{\varphi\theta}}^2 = \frac{{A_x}^2 + {A_y}^2}{{A_x}^2 + {A_y}^2 + {A_z}^2} \leq \frac{{A_x}^2}{{A_x}^2 + {A_z}^2} + \frac{{A_y}^2}{{A_y}^2 + {A_z}^2} = {e_\varphi}^2 + {e_\theta}^2. \tag{41}$$

Therefore, from Eq. (41), $e_{\varphi\theta}$ will also approach the region where Q_p is positive definite as e_φ and e_θ will diminish asymptotically because of the separated roll and pitch control. Also, by taking the derivative of Eqs. (14) and (20),

$$-\lambda_{max}\left(Q_\varphi\right)\|X_\varphi\|^2 \leq \dot{V}_\varphi \leq -\lambda_{min}\left(Q_\varphi\right)\|X_\varphi\|^2, \tag{42}$$

$$-\lambda_{max}(Q_\theta)\|X_\theta\|^2 \leq \dot{V}_\theta \leq -\lambda_{min}(Q_\theta)\|X_\theta\|^2, \tag{43}$$

$$X_\varphi = [e_\varphi \quad e_{\dot\varphi}]^T, \tag{44}$$

$$Q_\varphi = \begin{bmatrix} c_\varphi k_\varphi & \frac{1}{2} c_\varphi k_{\dot\varphi} \\ \frac{1}{2} c_\varphi k_{\dot\varphi} & k_{\dot\varphi} - c_\varphi J_{xx} \end{bmatrix}, \tag{45}$$

$$X_\theta = [e_\theta \quad e_{\dot\theta}]^T, \tag{46}$$

$$Q_\theta = \begin{bmatrix} c_\theta k_\theta & \frac{1}{2} c_\theta k_{\dot\theta} \\ \frac{1}{2} c_\theta k_{\dot\theta} & k_{\dot\theta} - c_\theta J_{yy} \end{bmatrix}. \tag{47}$$

Then, considering the combined position, roll and pitch controls,

$$V_{p\varphi\theta} = V_p + V_\varphi + V_\theta, \tag{48}$$

$$\begin{aligned} \dot V_{p\varphi\theta} \le & -[\|X_p\| \quad \|X_\varphi\|] \begin{bmatrix} \delta\lambda_{min}(Q_p) & -\frac{1}{2}\|Q_{p\varphi}\| \\ -\frac{1}{2}\|Q_{p\varphi}\| & \lambda_{min}(Q_\varphi) \end{bmatrix} \begin{bmatrix} \|X_p\| \\ \|X_\varphi\| \end{bmatrix} \\ & -[\|X_p\| \quad \|X_\theta\|] \begin{bmatrix} (1-\delta)\lambda_{min}(Q_p) & -\frac{1}{2}\|Q_{P\theta}\| \\ -\frac{1}{2}\|Q_{P\theta}\| & \lambda_{min}(Q_\theta) \end{bmatrix} \begin{bmatrix} \|X_p\| \\ \|X_\theta\| \end{bmatrix}, \end{aligned} \tag{49}$$

$$Q_{p\varphi} = \begin{bmatrix} mc_p(g + \|a_d\|) & 0 \\ m(g + \|a_d\|) & 0 \end{bmatrix}, \tag{50}$$

$$Q_{p\theta} = \begin{bmatrix} mc_p(g + \|a_d\|) & 0 \\ m(g + \|a_d\|) & 0 \end{bmatrix}, \tag{51}$$

$$0 < \delta < 1. \tag{52}$$

$V_{p\varphi\theta}$ is positive definite as c_p, c_φ and c_θ can be chosen to be arbitrarily small. Assuming $\|a_d\|$ is bounded, the combined position, roll and pitch control would be asymptotically stable as $\dot V_{p\varphi\theta}$ would be negative definite as $\lambda_{min}(Q_\varphi) > \frac{\|Q_{p\varphi}\|^2}{4\delta\lambda_{min}(Q_p)}$ and $\lambda_{min}(Q_\theta) > \frac{\|Q_{p\theta}\|^2}{4(1-\delta)\lambda_{min}(Q_p)}$, which requires k_φ, $k_{\dot\varphi}$, k_θ and $k_{\dot\theta}$ to be sufficiently large.

6.3. Proof of Asymptotic Stability for the Overall System

Since the separated yaw controller is asymptotically stable, the asymptotic stability of the overall system can be shown by adding V_ψ and $\dot V_\psi$ to $V_{p\varphi\theta}$ and $\dot V_{p\varphi\theta}$, respectively.

7. References

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