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Model-Free Sliding Mode Control for Coupled Square and Non-Square MIMO Systems

Aashrita Mandalapu[1*], Joshua Coleman[1*], Agamemnon Crassidis[1]
[1]Rochester Institute of Technology
1 Lomb Memorial Drive, Rochester, New York, 14623, USA
am2313@rit.edu; jcc2126@rit.edu; alceme@rit.edu
*These authors contributed equally to this work.

Abstract **–** Due to the correlation between increasing system complexity and performance requirements, the need for globally robust control systems is more and more apparent. This paper proposes two extensions to a novel globally robust control system called model free sliding mode control (MFSMC). First, a new approach to estimating the boundaries of the influence matrix without a system model is developed. Next, the use of hyper-plane transformations in MFSMC is discussed. Both techniques were individually paired with sliding mode controllers and simulated against a nonlinear test system. In the future, the methods will be added to MFSMC to broaden the applicability of this type of controller.

Keywords**:** Sliding Mode Control, Globally Robust Control, Estimation, Hyper-plane transformation, non-square MIMO system, Model Free Sliding Mode Control

1. Introduction

As technology advances, systems modelling has become increasingly difficult. Further, today's complicated, high performing systems contain severe nonlinearities. As a result, traditional control methods such as PID control are often used at the cost of efficiency and performance. Modern Lyapunov-based controllers are more robust, which is important when stability is desired even with modelling assumptions and uncertainties. In many cases, however, the amount of uncertainties makes using most Lyapunov-based controllers impractical. For example, a sliding mode controller used to control a system with high uncertainties would require a large controller gain, and, as a result, a high control output.

In order to feasibly control these systems, multiple globally robust controllers have been developed. These controllers generally use two methods to achieve stability no matter the uncertainty: limiting the form of the error in the system or estimating the maximum uncertainty. One example of the first method was given in [1]. This LQR-based controller was proven to cause stability so long as the uncertainties were in the image of the control influence matrix. On the other hand, the controller in [2] would cause stability when a certain parameter was larger than a linear combination of the uncertainty and its derivative. An estimator was created to keep the parameter above the combination. Both controllers were tested and performed well but are limited in their scope (linear systems for [1] and second-order SISO systems for [2]). These two controllers illustrate the need for a broadly applicable globally robust controller; such a controller would guarantee stability no matter the system form.

A controller with the potential to be broadly globally robust was introduced in [3] and developed in [4] and [5]. This controller method – known as Model Free Sliding Mode Control (MFSMC) – solved the issue of large uncertainties in the system by avoiding a system model entirely. Instead, it relied on the following unity gain equation and assumptions:

$$\vec{\mathrm{x}}^{(n)} = \vec{\mathrm{x}}^{(n)} + B[\vec{\mathrm{u}} - \vec{\mathrm{u}}_{k-1}] + \vec{\varepsilon} \tag{1}$$
$$\vec{\varepsilon} = B[\vec{\mathrm{u}}_{k-1} - \vec{\mathrm{u}}] \tag{2}$$
$$\hat{\varepsilon} = B[\vec{\mathrm{u}}_{k-2} - \vec{\mathrm{u}}_{k-1}] \tag{3}$$
$$|\vec{\varepsilon}| < (1+\sigma_u)|\hat{\varepsilon}| \tag{4}$$
$$B_{min} < B < B_{max} \tag{5}$$

Sliding mode control techniques are then applied. The resultant controller is as follows:

$$\vec{u} = \hat{B}^{-1}\left[-\tilde{x} - \hat{\varepsilon} - \left(\dot{\vec{s}} - \tilde{x}^{(n)}\right) - \vec{\kappa} \circ sgn(\vec{s})\right] \tag{6}$$

$$\vec{\kappa} = |\beta - I|\left(\left|\tilde{x}^{(n)}\right| + \left|\bar{L}\dot{\tilde{x}}\right|\right) + |\beta(1+\sigma_u) - I||\hat{\varepsilon}| + \beta\vec{\eta} \tag{7}$$

A boundary layer $\vec{\Phi}$ was also introduced to reduce dithering. A derivation in a sliding mode context is given in [6].

These papers demonstrated that MFSMC was applicable to all square systems (systems in which the number of control inputs is equal to the number of non-derivative states) so long as the bounds of the influence matrix were known. [7] proposed the use of an estimator for the parameter $\vec{\eta}$, which would avoid the need to know the bounds. Due to efficiency concerns, [8] adapted the estimator to find the matrix directly. While the controller-estimator combination stabilized several test systems, the estimator's performance was sub-par in some circumstances.

Currently work is being performed to solve both limitations. In this work, both a method for estimating the bounds on the influence matrix and the use of a hyper-plane transformation for the control of non-square systems are presented. Results of using these techniques in a sliding mode context are given. In the future, the methods will be adapted to MFSMC.

2. Boundary Estimation

Using a traditional estimator in MFSMC is more difficult than it may first seem – without a model, finding a suitable regression equation is not straightforward. The method in [8] was adapted from least squares with bounded gain forgetting. Rather than deriving the estimator from a relationship involving the influence matrix, [8] created a least squares with bounded gain forgetting estimator using the sliding condition. This condition was developed as a part of sliding mode control:

$$\left(\dot{\vec{\Phi}} - \vec{\eta}\right) \circ |\vec{s}| \geq \vec{s} \circ \dot{\vec{s}} \tag{8}$$

$$\vec{s} = \left[\frac{d}{dt} + \Lambda\right]^{n-1} \tilde{x} \tag{9}$$

where $\tilde{x}$ is the difference between the current states $\vec{x}$ and their desired values $\vec{x}_d$ and "∘" is an elementwise product. When Eq. (8) is satisfied, the system is stable. The estimate is multiplied by a factor to get the matrix's bounds.

The new technique similarly uses the sliding condition. Given a system of the form

$$\vec{x}^{(n)} = \vec{f}\left(\vec{x}^{(i)}, t\right) + B\left(\vec{x}^{(i)}, t\right)\vec{u} \tag{10}$$

the estimator's goal is to find the influence matrix bounds. Start by substituting Eq. (10) into Eq. (9) and rearranging to get

$$sgn(\vec{s}) \circ B\vec{u} \geq sgn(\vec{s}) \circ \left[\vec{x}_d^{(n)} - \vec{f} - \left(\dot{\vec{s}} - \tilde{x}^{(n)}\right)\right] + \left(\dot{\vec{\Phi}} - \vec{\eta}\right) \tag{11}$$

Next, multiply both sides by a factor $\vec{\alpha}$ which is 1 when the sliding condition is not satisfied and -1 in all other situations. Note doing so flips the inequality. The factor generalizes the inequality to all states of the sliding condition.

At this point, the derivation diverges for coupled and decoupled systems. First, consider the case when the system is decoupled (B is diagonal). If $\vec{\mathrm{b}}$ is a vector of the diagonal elements of B ($B = diag(\vec{\mathrm{b}})$), the left side reduces to

$$\vec{\alpha} \circ sgn(\vec{s}) \circ B\vec{u} = \vec{\alpha} \circ sgn(\vec{s}) \circ \vec{b} \circ \vec{u} \ = diag(\vec{\alpha} \circ sgn(\vec{s}) \circ \vec{u})\vec{b} \tag{12}$$

The final step is to define the following values:

$$A = |diag(\vec{\alpha} \circ sgn(\vec{s}) \circ \vec{u})| \tag{13}$$

$$\vec{q} = \vec{\alpha} \circ \left(\left| \dot{\vec{\Phi}} - \vec{\eta} + sgn(\vec{s}) \circ \left(\vec{x}_d^{(n)} - \overline{L}\dot{\vec{x}} \right) \right| + \overline{f} \circ (sgn(\vec{s}))^2 \right) \tag{14}$$

$$\overline{f} = \begin{cases} min\left(|\vec{f}_{min}|, |\vec{f}_{max}|\right), & B_{min} \ is \ being \ estimated \\ max\left(|\vec{f}_{min}|, |\vec{f}_{max}|\right), & B_{max} \ is \ being \ estimated \end{cases} \tag{15}$$

Using these definitions, the influence matrix bounds may be estimated as

$$B_{k|k} = max\left(B_{k|k-1}, B_{k-1|k-1}\right) \tag{16}$$

$$B_{k|k-1} = diag(A^{+}\vec{q}) \tag{17}$$

Where A^{+} is the pseudoinverse of A and $B_{k|k}$ is the current bounds estimate. $B_{k|k-1}$ is the exact solution to the equation. However, if any of the current values of B are less than their corresponding maxima, some of the estimated values will be too low. For that reason, the actual estimate $B_{k|k}$ is found by comparing the equation's solution with the last estimates. A similar approach may be taken to derive an estimator for coupled systems. In testing, these definitions – which are derived from the maximizations of each side of the inequality – performed better than alternatives.

3. Hyper-plane Transformation

Using the MFSMC approach perfect tracking and stability for square MIMO systems was obtained but additional methods are needed to guarantee tracking for underactuated (non-square) MIMO systems. A challenge in dealing with non-square systems is the non-invertibility of the input gain matrix, making the formulation of the control law impossible. To address this issue, a potential solution involves employing a coordinate transformation on the system. Through this transformation, the originally "non-square" matrix can effectively be "squared," overcoming the non-invertibility limitation and facilitating the formulation of the control law. This method being implemented in this work is the hyperplane transformation.

Consider the following n^{th}-order autonomous system:

$$x_p^n = f_p(x) + [B]_{p\times m} u_m \tag{18}$$

where $m < p$, and the matrix $[B]$ is non-square. Let:

$$\vec{y} = [T]\vec{x} \tag{19}$$

where the dimensions of matrix $[T] =$ the dimensions of $[B]'$. The above can be rewritten as:

$$y_p^{(n)} = [T]_{m\times p} f_p(x) + \left[[T]_{m\times p}[B]_{p\times m}\right] u_m \tag{20}$$

and the product of $[[T][B]]$ is now square and invertible. The matrix $[T]$ can be thought of as a weighing matrix. Since the system under consideration is underactuated, and states cannot display perfect tracking simultaneously, $[T]$ can be used to track certain outputs "more heavily" than others.

To apply the model-free SMC method to an underactuated MIMO system, knowledge of the size of the $[B]$ matrix of the system is required to formulate the transformation matrix $[T]$. Once that is known, the model-free SMC scheme is developed in the y coordinate system, in a similar manner to the derivation in square MIMO systems, and $[T]$ is used to relate y to x, and vice versa.

4. Results

4.1. Simulation Results for Boundary Estimator SMC

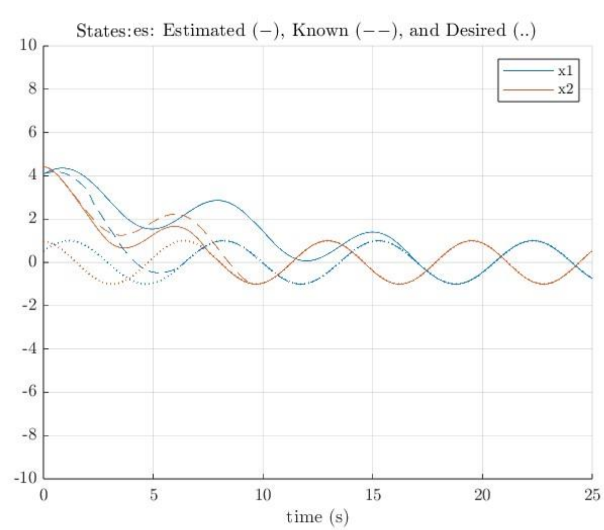

Fig. 2: Simulated results for SMC with and without estaimtion against desired states (solid, dashed, and dotted, respectively).

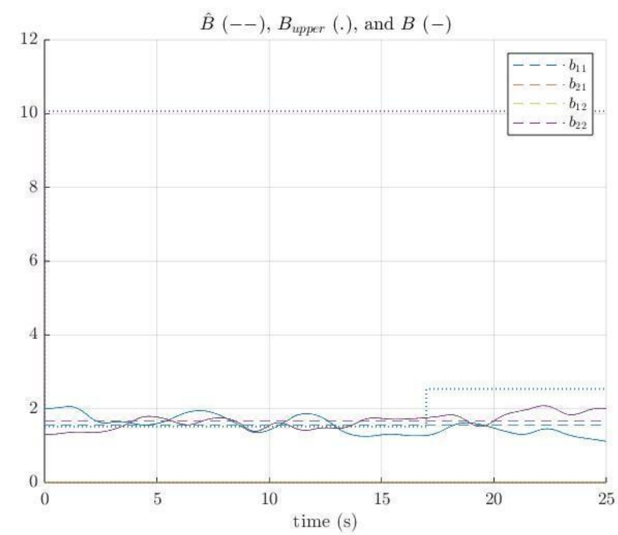

Fig. 1: Influence matrix boundary estimates (dotted) along with its estimated (dashed) and actual (solid) values.

Simulated results for SMC with and without boundary estimation for the system

$$\ddot{x}_1 = -a_1(t)\dot{x}_1{}^2\cos(2x_1)\,x_2 + b_{11}(t)u_1 \tag{21}$$

$$\ddot{x}_2 = -a_2(t)\dot{x}_2{}^2\dot{x}_1x_2 + b_{22}(t)u_2 \tag{22}$$

are given in Figs. 1 and 2. Unsurprisingly, the controller with known bounds reaches the desired states faster overall (though state 2 of the estimated controller beats that of the regular SMC). This discrepancy is due to the time spent estimating the boundaries. Fig. 2 shows the boundary estimates are above the maximum values. While the amount of estimation would seem to cause inefficiencies, Fig. 3 indicates the estimated SMC has a lower control output.

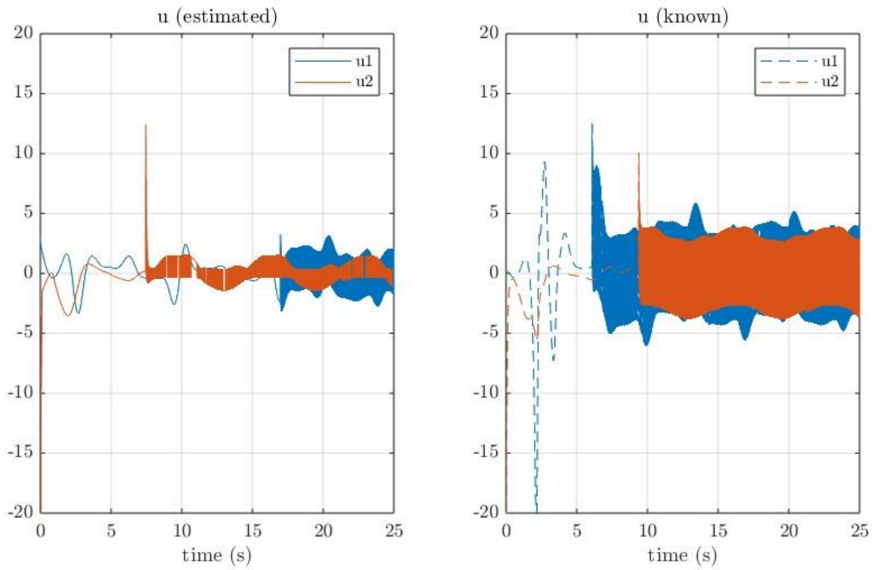

Fig. 3: Control output for SMC with estimated (left) and known (right) influence matrix boundaries.

Figs. 4 and 5 show the SMC with boundary estimation's performance, this time against a coupled system. In Fig. 5, the estimator is giving values of δ, a parameter related to B by

$$\delta = B\hat{B}^{-1} - I \tag{23}$$

The test system is the same as in Eqs. (21) and (22), except with $b_{12}(t)$ and $b_{21}(t)$ terms. Again, state 1 is sluggish, but the system is still stable.

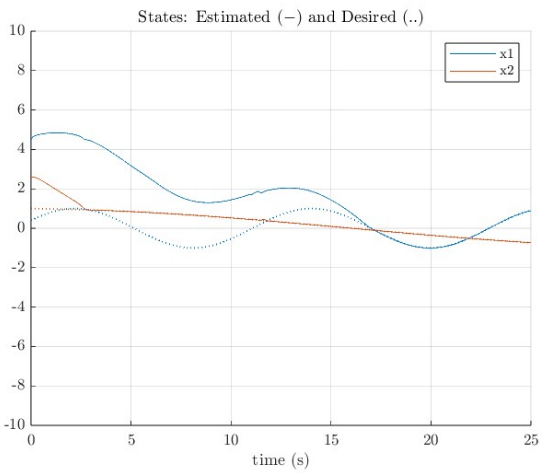

Fig. 4: Desired (dotted) and simulated (solid) state trajectories for SMC with boundary estimation on a coupled system.

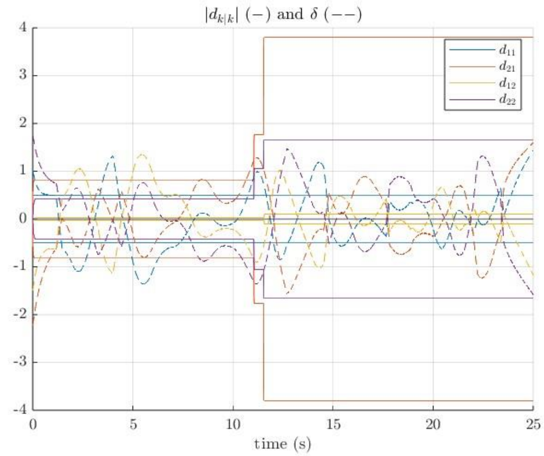

Fig. 5: Estimated delta matrix bounds (solid) against their actual values (dashed).

4.2. Simulation Results for Hyper Plane Transformation in SMC

Simulated results for traditional SMC with hyperplane transformation including boundary layer for the following system:

$$\ddot{x}_1 + a_1(t)[x_1 + \dot{x}_1]\, cos\,(x_2) \;= u(t) \tag{24}$$

$$\ddot{x}_2 + a_2(t)[x_2 + \dot{x}_2 - \dot{x}_1]|x_1| = 0 \tag{25}$$

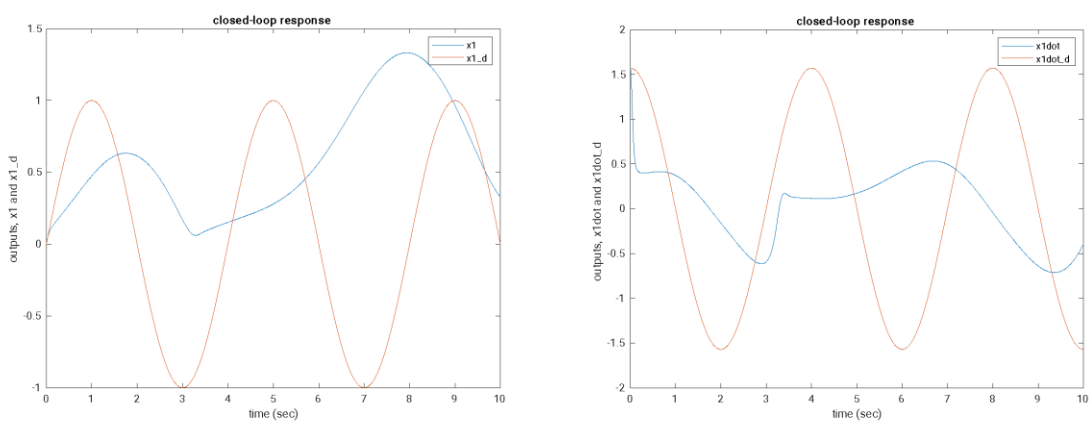

Fig. 6: Closed loop response of x_1 vs x_{1_d} and $\dot{x_1}$ vs $\dot{x_1}_d$

It can be seen from Figs. 6 and 7 that the tracking of x_1 and $\dot{x_1}$ with the desired signals is not so great but there is close to perfect tracking for x_2 and $\dot{x_2}$. This is because of the values set in the transformation matrix [T]. x_2 and $\dot{x_2}$ is more heavily tracked than x_1 and $\dot{x_1}$. Based on the requirement the [T] matrix can be set to either track x_1 and $\dot{x_1}$ or x_2 and $\dot{x_2}$.

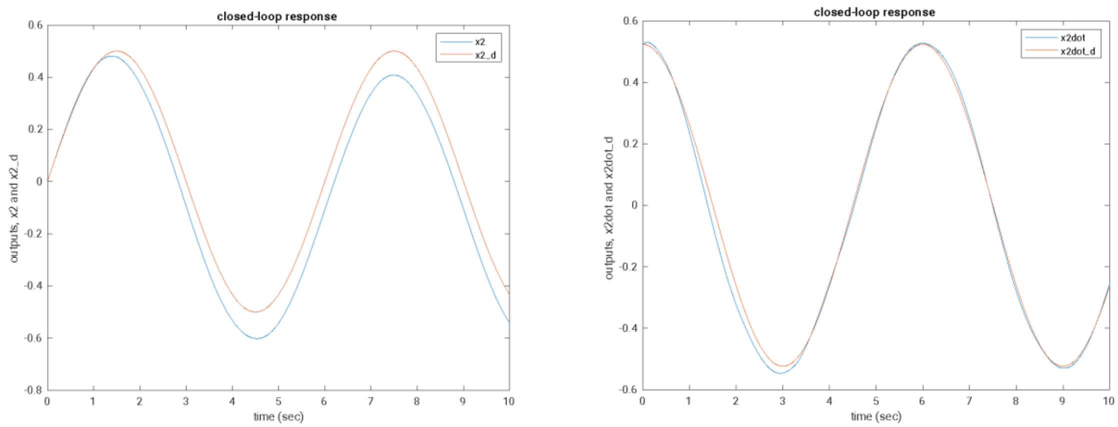

Fig. 7: Closed loop response of $x_2\ vs\ x_{2_d}\ and\ \dot{x}_2\ vs\ \dot{x}_{2_d}$

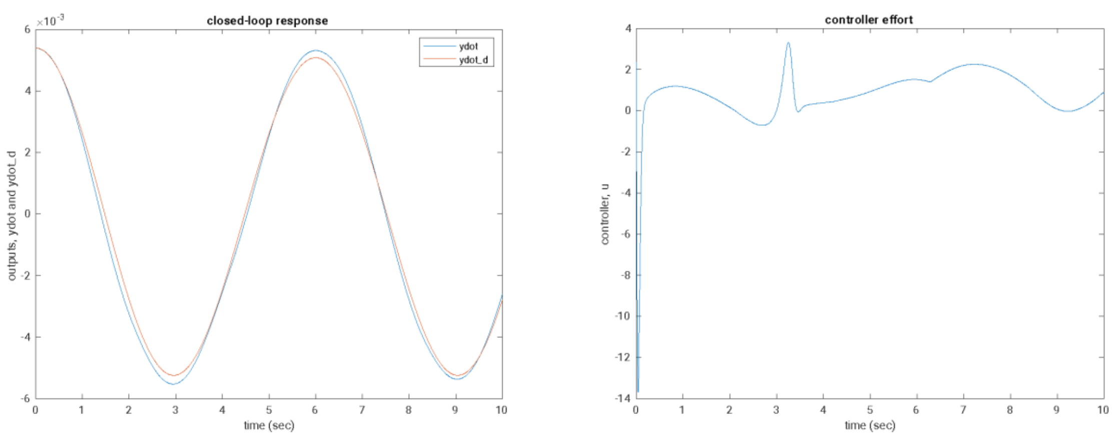

Fig. 8: Closed loop response of $\dot{y}\ vs\ \dot{y_d}$ and the controller effort "u"

While the original system's signal tracking is based on the [T] matrix, it can be seen from the left side of Fig. 8 that the transformed systems signal tracking is close to its desired value. The controller effort for the system is displayed on the right of Fig. 8.

Left side of Fig. 9 shows that the system's sliding condition is satisfied which proves that the system is stable. The S-trajectories lie within the defined boundary layer which can be seen on the right side of Fig. 9.

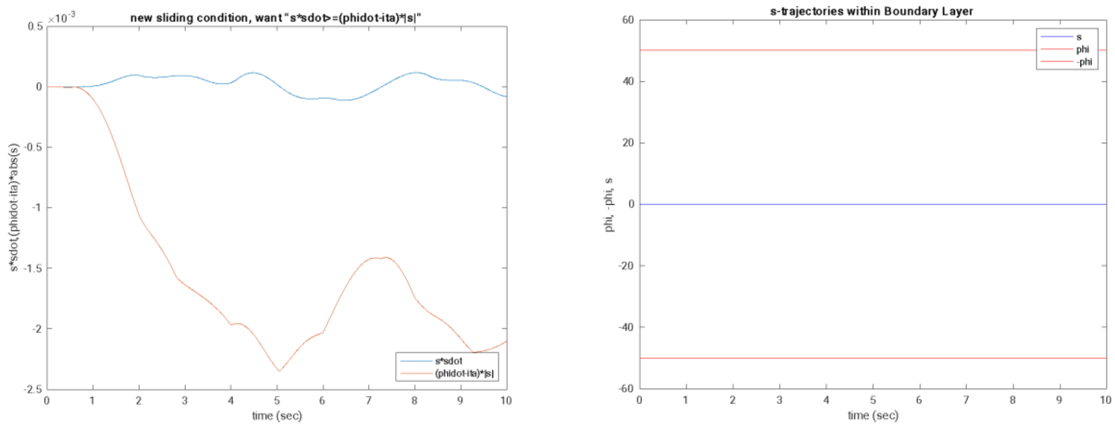

Fig. 9: Sliding and boundary layer condition satisfied.

5. Conclusion

This paper introduced two methods: One, the integration of MFSMC approach with an online parameter (boundary) estimation method for complex nonlinear systems which allows for dynamic updates to the control law based on evolving system characteristics. The introduced approach includes a boundary layer to limit chattering and has precise tracking along with proven stability. The second method is the integration of hyperplane transformation with a traditional SMC for non-square MIMO systems with the inclusion of boundary layer. The tracking in this method depends on the values chosen in the transformation matrix. The next steps would be to apply both methods to model free control and evaluate their performance. Upon obtaining the results for model free control then this type of control approach can be applied to any physical systems making it more robust and stable.

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