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Stabilization of a Class of Discrete-Time Nonlinear Stochastic Systems Using Static Output Feedback

Justin J. Kennah, Edwin E. Yaz, Susan C. Schneider
Marquette University
1515 W. Wisconsin Ave., Milwaukee, USA
justin.kennah@marquette.edu; edwin.yaz@marquette.edu; susan.schneider@marquette.edu

Abstract **-** In this paper, static output feedback control is proposed to stabilize a general class of discrete-time stochastic nonlinear systems. Knowledge of the precise form of the nonlinearity or its statistics are not required. Instead, it is only necessary that a bound on the second moment of nonlinearity can be determined. The control gain is determined by solving a linear matrix inequality which is sufficient to show that the controlled system is stable in the mean square and almost sure senses.

Keywords**:** Output Feedback; Nonlinear systems; Stochastic systems; Discrete-time systems; Linear matrix inequality

1. Introduction

This paper considers static output feedback stabilizing control for a general class of discrete-time nonlinear stochastic systems given by the following equations

$$x_{k+1} = Ax_k + Bu_k + f(x_k, u_k, w_k) \tag{1}$$

$$y_k = Cx_k \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the input, $w_k \in \mathbb{R}^p$ is an independent zero mean noise sequence, $y_k \in \mathbb{R}^q$ is the output, and $f(x_k, u_k, w_k): \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ is a nonlinear function satisfying the following properties

$$f(0,0,w_k) = 0 \tag{3}$$

$$E_{x_k}\{f(x_k, u_k, w_k)\} = 0 \tag{4}$$

$$E_{x_k}\{f(x_k, u_k, w_k) f^{\mathrm{T}}(x_j, u_j, w_j)\} = 0 \;\forall k \neq j \tag{5}$$

$$E_{x_k}\{f(x_k, u_k, w_k) f^{\mathrm{T}}(x_k, u_k, w_k)\} \leq \sum_{i=1}^{r} T^i \left(x_k^{\mathrm{T}} M^i x_k + u_k^{\mathrm{T}} N^i u_k\right) \tag{6}$$

where $E_{x_k}\{f\}$ denotes the expectation of f conditional on x_k, and $r = n(n+1)/2$. Additionally, the matrix bounds $T^i, M^i \in \mathbb{R}^{n\times n}$, and $N^i \in \mathbb{R}^{m\times m}$, are known for all i. Further, since Eqn. (6) represents the upper bound of a covariance relation, it follows that all matrices on the right-hand side of the inequality are symmetric and at least positive semidefinite.

This particular class of systems, as pointed out in [1]-[4], is in fact quite general and includes several well-known systems such as linear systems with additive noise, linear systems with state and control multiplicative noise, state and control norm dependent random vectors, random vector dependent on the sign of a nonlinear function of the state, and many others.

Several researchers have investigated this particular class of systems in the past. It was shown in [1] that the optimal finite horizon controller which minimizes a quadratic performance criteria, is a linear function of the states of the system. However, this method assumes perfect knowledge of the states of the system. Additionally, the form of the nonlinearity is assumed to be known requiring that the inequality given in Eqn. (6) is a strict equality. Further, determination of the control law relies on the solution to a backward running generalized Riccati equation which must be solved offline. A suboptimal version of the work in [1] was found in [2]. The benefit of the suboptimal version is that the control law is a constant state variable feedback gain. However, perfect knowledge of the states, as well as the form of the nonlinearity, are again required. The work in [3] proposed an optimal infinite horizon optimal control which also required all states to be measurable and exact form of the nonlinearity to be known. Control of systems with incomplete state knowledge was considered in [4] and a reduced-order observer was proposed in [5]. Design of several linear state estimators using various performance criteria was considered in [6]. The work in [7] considered a general performance criteria which was minimized to achieve several control objectives including H_2, H_∞, and several passivity results. This work relaxed the condition that the exact form of the nonlinearity be known and required that only knowledge of the upper bound of the covariance of the nonlinearity be known (similar to Eqn. (6)). However, similar to the majority of the previously listed works, the work in [7] also required all state variables to be measurable.

The method proposed in this paper extends the previous work by considering a static gain output feedback controller which has not been considered until now. The benefits of the proposed approach are: 1) the exact form of the nonlinearity need not be known, rather it is only necessary that an upper bound on the second moment of the nonlinear stochastic term can be determined, 2) complete knowledge of the states is not needed, 3) there is no need to design an estimator in the event that perfect knowledge of the states is unavailable.

The following additional notation is used in this paper; $\|x_k\|$ is used to denote the Euclidean vector norm of x_k, $E\{x_k\}$ is the unconditional expectation of x_k, $P > 0$ $(P < 0)$ and $P \geq 0$ $(P \leq 0)$ are used to respectively denote a positive (negative) definite and positive (negative) semidefinite matrix P. The trace of the matrix T is given by $\mathrm{Tr}[T]$. Respectively, I_n and $[0]_{n\times m}$ represent an n-dimensional identity matrix and a $n\times m$ null matrix. The square root (square root transpose) of matrix N is given by $N^{1/2}$ $\left(N^{T/2}\right)$.

This paper uses Schur's complement frequently which is stated as follows; given matrices A, B, and C of appropriate dimensions, the following statements are equivalent:

$$i) \quad \begin{bmatrix} A & B \\ B^{\mathrm{T}} & C \end{bmatrix} > 0$$

$$ii) \quad A - BC^{-1}B^T > 0, \qquad C > 0$$

$$iii) \quad C - B^T A^{-1} B > 0, \qquad A > 0$$

2. Main Result

Let the control be given by

$$u_k = Ky_k = KCx_k \tag{7}$$

where $K \in \mathbb{R}^{q\times m}$ is a static gain which is to be determined. By substituting Eqn. (7) into Eqns. (1) and (6), the system equation and the upper bound on the covariance of the nonlinearity are given respectively as

$$x_{k+1} = (A + BKC)x_k + f(x_k, u_k, w_k) \tag{8}$$

$$E_{x_k}\{f(x_k, u_k, w_k)f^{\mathrm{T}}(x_k, u_k, w_k)\} \leq \sum_{i=1}^{r} T^i x_k^{\mathrm{T}}\left(M^i + C^T K^T N^i KC\right)x_k \tag{9}$$

Before proceeding further, several definitions, and concepts, related to mean square and almost sure stochastic stability are reviewed.

Definition 1: The system is said to be *mean square stable* if for any initial state, x_0, $\sup_k E\{\|x_k\|^2\} < \infty$ holds for all for all k.

Definition 2: The system is said to be *mean square asymptotically stable* if the system is mean square stable, and and $E\{\|x_k\|^2\} \to 0$ as $k \to \infty$ for any initial state, x_0.

Definition 3: The system is said to be *almost surely stable* if for any initial state, x_0, $\|x_k\|^2 < \infty$ holds for all k with probability equal to 1.

Definition 4: The system is said to be *almost surely asymptotically stable* if the system is almost surely stable and $\|x_k\|^2 \to 0$ as $k \to \infty$ with probability equal to 1 for any initial state, x_0.

Defining the Lyapunov function $V_k(x_k) = x_k^{\mathrm{T}} P x_k$ for some unknown $P = P^{\mathrm{T}} > 0$, it follows that the system in Eqn. (8) will be asymptotically stable in the mean square and almost sure senses if the following condition is satisfied [3]

$$E_{x_k}\{V_{k+1}(x_{k+1}) - V_k(x_k)\} = E_{x_k}\{x_{k+1}^{\mathrm{T}} P x_{k+1} - x_k^{\mathrm{T}} P x_k\} < 0 \tag{10}$$

Substituting for x_{k+1} and using the relation in Eqn. (4) allows us to express Eqn. (10) as

$$x_k^{\mathrm{T}}[(A + BKC)^{\mathrm{T}} P (A + BKC) - P] x_k + E_{x_k}\{f^{\mathrm{T}}(x_k, u_k, w_k)\, P\, f(x_k, u_k, w_k)\} < 0 \tag{11}$$

Using the properties of the trace operator and substituting Eqn. (9) into the above inequality yields

$$x_k^{\mathrm{T}}\left[(A + BKC)^{\mathrm{T}} P (A + BKC) - P \ + \sum_{i=1}^{r} \mathrm{Tr}[PT^i]\left(M^i + C^T K^T N^i K C\right)\right] x_k < 0 \tag{12}$$

which implies

$$P > \ (A + BKC)^{\mathrm{T}} P (A + BKC) \ + \sum_{i=1}^{r} \mathrm{Tr}[PT^i]\left(M^i + C^T K^T N^i K C\right) \tag{13}$$

Thus, Eqn. (13) is a sufficient condition to ensure the mean square and almost sure asymptotic stability of the system.

A computational problem arises when attempting to construct a Linear Matrix Inequality (LMI) from the above nonlinear matrix inequality due to the unknowns P and K appearing in a nonlinear form. This can be overcome by considering the boundedness and positive definiteness of P as follows:

$$\gamma I_n \geq P > 0 \tag{14}$$

with

$$\gamma \in \mathbb{R}, \qquad \gamma > 0 \tag{15}$$

Substituting the bounds for P into Eqn. (13) to find a sufficient condition for the inequality to hold, we can write

$$P \ > \ \gamma (A + BKC)^{\mathrm{T}} (A + BKC) \ + \gamma \sum_{i=1}^{r} \mathrm{Tr}[T^i]\left(M^i + C^T K^T N^i K C\right) \tag{16}$$

Dividing both sides by γ we obtain

$$\tilde{P} > (A + BKC)^{\mathrm{T}}(A + BKC) + \sum_{i=1}^{r} \mathrm{Tr}[T^i]\left(M^i + C^T K^T N^i KC\right) \tag{17}$$

where

$$\tilde{P} = \frac{1}{\gamma}P, \qquad \tilde{P} > 0 \tag{18}$$

the inequality follows

$$\tilde{P} - (A + BKC)^{\mathrm{T}}(A + BKC) - \sum_{i=1}^{r} \mathrm{Tr}[T^i]\left(M^i + C^T K^T N^i KC\right) > 0 \tag{19}$$

Using Schur's complement we can express the above matrix inequality as

$$\begin{bmatrix} \tilde{P} - \sum_{i=1}^{r} \mathrm{Tr}[T^i]\left(M^i + C^T K^T N^i KC\right) & (A + BKC)^{\mathrm{T}} \\ A + BKC & I_n \end{bmatrix} > 0 \tag{20}$$

Then, by using Schur's complement on the 1,1 block element of the above inequality, the following LMI in $\tilde{P}$ and K is obtained

$$\begin{bmatrix} \tilde{P} - \sum_{i=1}^{r} \mathrm{Tr}[T^i] M^i & (A + BKC)^{\mathrm{T}} & C^T K^T \left(\sum_{i=1}^{r} \mathrm{Tr}[T^i] N^i\right)^{\frac{T}{2}} \\ A + BKC & I_n & [0]_{n\times m} \\ \left(\sum_{i=1}^{r} \mathrm{Tr}[T^i] N^i\right)^{\frac{1}{2}} KC & [0]_{m\times n} & I_m \end{bmatrix} > 0 \tag{21}$$

In summary, by using the definitions of mean square and almost sure asymptotic stability we can conclude that the general nonlinear stochastic system given by Eqns. (8) and (9) can be stabilized, in the mean square and almost sure senses, using a static output feedback gain, K, which can be obtained by solving the LMI in Eqn. (21) with the restriction given in Eqn. (18).

3. Simulation Study

In this section, Chua's circuit [8] is used to demonstrate controller design under the proposed method. Chua's circuit is a well-known nonlinear circuit that elicits a chaotic response. The continuous-time equations governing Chua's circuit can be given by the state space model shown below [8].

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}-\alpha & \alpha & 0\\1 & -1 & 1\\0 & -\beta & -\mu\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}1\\1\\1\end{bmatrix}u+\begin{bmatrix}-\alpha f(x_1)\\0\\0\end{bmatrix} \tag{22}$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \tag{23}$$

where

$$\alpha = 10.0063, \qquad \beta = 16.5811, \qquad \mu = 0.138083 \tag{24}$$

are system parameters [9], and

$$f(x_1) = bx_1 + 0.5(a-b)(|x_1+1| - |x_1-1|) \tag{25}$$

is a nonlinear function in x_1 with

$$a = -1.39386, \qquad b = -0.75590 \tag{26}$$

Equivalently, the above nonlinearity can be piecewise defined as

$$f(x_1)=\begin{cases} bx_1+(b-a), & x_1 \le -1\\ ax_1, & |x_1|<1\\ bx_1+(a-b), & x_1 \ge 1\end{cases} \tag{27}$$

It is noted that the form of the nonlinearity used in this example is known. The decision to use a nonlinearity with a known form in this example was deliberate and necessary for the purposes of simulation. In general, it is not a requirement that the form of the nonlinearity be known. Rather, it is only required that a bound on the second moment of the nonlinearity can be determined.

The continuous-time system is discretized using forward Euler discretization with a time step of $T_s = 0.01s$ to yield the following discrete-time equations

$$\begin{bmatrix}x_{1,k+1}\\x_{2,k+1}\\x_{3,k+1}\end{bmatrix}=\begin{bmatrix}0.8999 & 0.1001 & 0\\0.0100 & 0.9900 & 0.0100\\0 & -0.1658 & 0.9986\end{bmatrix}\begin{bmatrix}x_{1,k}\\x_{2,k}\\x_{3,k}\end{bmatrix}+\begin{bmatrix}0.01\\0.01\\0.01\end{bmatrix}u_k+\begin{bmatrix}-0.1001f(x_{1,k})\\0\\0\end{bmatrix} \tag{28}$$

In this formulation, the nonlinear function, $f(x_{1,k})$, is treated as a stochastic nonlinearity. Therefore, the conditional covariance of the nonlinearity can be expressed as

$$E_{x_k}\left\{\begin{bmatrix}-0.1001f(x_{1,k})\\0\\0\end{bmatrix}\begin{bmatrix}-0.1001f(x_{1,k}) & 0 & 0\end{bmatrix}\right\}=\begin{bmatrix}0.0100f^2(x_{1,k}) & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} \tag{29}$$

Since $|a|>|b|$, it follows from Eqn. (27) that

$$\sup_{x_1} f^2(x_{1,k}) = \left(a x_{1,k}\right)^2 \tag{30}$$

which, from Eqn. (9), implies that the covariance of the nonlinearity can be bounded above by

$$\sum_{i=1}^{r} T^i\left(x_k^{\mathrm{T}} M^i x_k + u_k^{\mathrm{T}} N^i u_k\right) \leq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_k^{\mathrm{T}} \begin{bmatrix} 0.0100a^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_k \tag{31}$$

Thus, it follows that Eqn. (31) can be satisfied by allowing

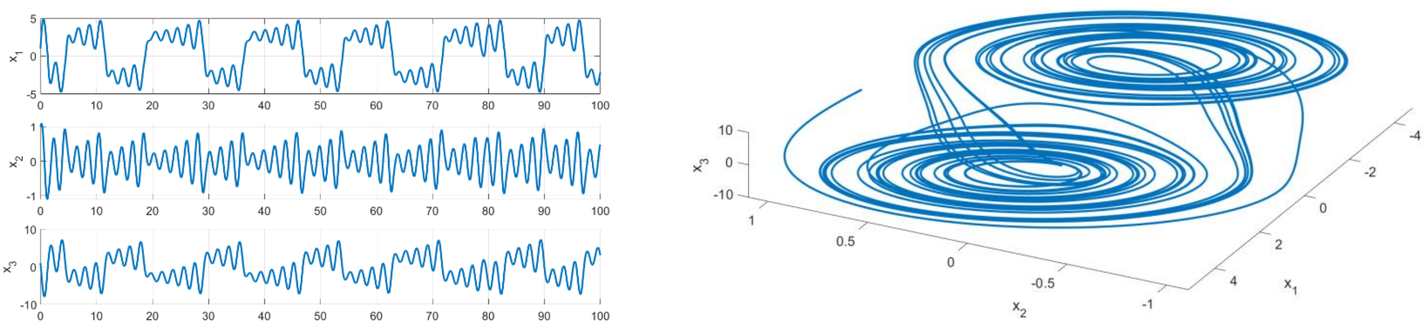
$$\sum_{i=1}^{r} T^i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \sum_{i=1}^{r} M^i = \begin{bmatrix} 0.0100a^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad N^i = 0 \; \forall i \tag{32}$$

Having determined an upper bound to the second moment of the nonlinearity, the LMI defined in Eqn. (21) can now be solved for $\tilde{P}$ and K, with the restriction given in Eqn (18). A solution to the LMIs which provides a stabilizing control is found to be

$$\tilde{P} = \begin{bmatrix} 2.2171 & 0 & 0 \\ 0 & 2.2074 & 0 \\ 0 & 0 & 2.2074 \end{bmatrix} \tag{33}$$

$$K = -30.3312 \tag{34}$$

Thus, the control given by $u_k = Ky$ will asymptotically stabilize the system in the mean square and almost sure senses. A simulation of the uncontrolled system is shown in Fig. 1. Figure 1(a) shows the evolution of the states of the system which appear to be periodic. Figure 1(b) shows the phase portrait of the system where the chaotic nature of the uncontrolled circuit is evident by observing that the trajectories never overlap. It is noted that the initial conditions for the simulation were chosen to be $x_0 = [1 \quad 1 \quad 1]^T$.



(a) Evolution of system states

(b) Chaotic orbit of the system states

Fig. 1: Simulation of uncontrolled Chua's circuit in discrete-time

A simulation of the controlled system is shown in Fig. 2 where it is shown that the output gain found in Eqn. (34) stabilizes the system.

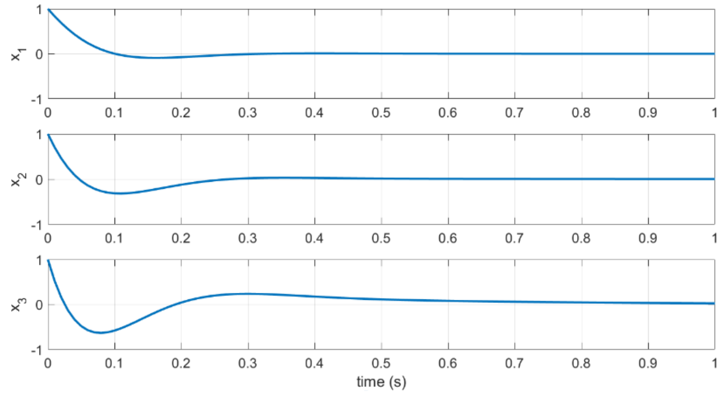


Fig. 2: Simulation of controlled Chua's circuit in discrete-time using output gain

4. Conclusion

In this work, a method to stabilize a class of discrete-time stochastically nonlinear systems using a static output feedback is proposed. Since the gain is applied directly to the output of the system, there is no need to have complete knowledge of the states of the system, nor is it necessary to design an observer to estimate the unknown states. Rather, only knowledge of a bound on the covariance of the nonlinear stochastic term is sufficient to determine a stabilizing control law for the system. A controller was designed to stabilize Chua's circuit under the proposed methodology and simulation results showed that the controller indeed stabilized the system as desired.

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