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Multicriteria State Estimate Feedback PID Controller with Regional Eigenvalue Assignment via Linear Matrix Inequalities

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Abstract **–** A state feedback PID controller is introduced where the gains are determined through the LMI methodology, which allows satisfaction of multiple performance criteria. Regional eigenvalue assignment with multiple regions is accomplished through time-domain design criteria such as rise time, percent overshoot, and settling time. The H_2 criterion is also satisfied which adds optimality to the solution. Feedback gains are calculated off-line and all tuning is eliminated The theoretical results are verified through MATLAB simulations to confirm that the required criteria are met.

Keywords**:** State feedback control, proportional integral derivative control, robustness, linear matrix inequalities

1. Introduction

PID control is the most widely used technique in industrial applications [1], which is attributed to its robust performance and functional simplicity in many different systems. The control of systems within the large set of multiple input multiple output (MIMO) PID problems has been documented thoroughly with some methods developed that not only accomplish control of the system but do so robustly [2]. There are also methods that automate the tuning of PID controllers via solving iterative LMI [3]. While a MIMO PID controller with fully automated tuning via LMI has already been designed, the iterative nature adds significant computational complexity. In this paper a new LMI design method for MIMO PID controllers that is easier to implement in industrial situations is developed.

Regional Eigenvalue Assignment (REA) allows for the combination of different time-domain criteria that relate to the regional location of a systems poles. REA has been used in many different applications from state and output feedback [4] and in combination with optimization conditions such as H_2 [5] and $H\infty$ [6]. Combining REA and H_2 to find PID gains that hold the regional information while also meeting robust criteria metrics is the core idea of this work. Because of the problems that arise from using derivatives in a control system, a reduced order observer is constructed in order to estimate the derivative states needed for full state feedback.

The solution provided in this work is a state estimate feedback PID controller with control gains found through the solutions of LMIs that could control system robustly while remaining practical for industrial solutions with PID controllers. The method can be applied for multivariable systems or even a linearized non-linear system defined at a set point. To estimate any unknown states, a reduced order observer is also designed. A reduced order observer is used as it often performs better than a full-order observer in a closed-loop control system as it does not redundantly estimate known state variables and requires less computations [7].

2. Problem Formulation

For the following 2nd order continuous linear time invariant system

$$M\ddot{x} + D\dot{x} + Rx = u \tag{1}$$

where $x \in R^n$ is the state of the system and $u \in R^m$ is the input to the system, a PID controller with proportional, integral, and derivative gains used as the control input. For PID control, an additional state variable, z, is introduced to account for the integral portion of the control.

$$z = \int_0^t x \, d\tau \tag{2}$$

The integral of the state, z, can be constructed easily from x or $\hat{x}$; $\dot{x}$ is estimated through a reduced order observer. With the added integral term, the PID control input of the system is defined as,

$$u = Kx = K_p x + K_d \dot{x} + K_i z. \tag{3}$$

The system described by (1) is put into its state space equivalent.

$$\ddot{x} = (-M^{-1}R)x + (-M^{-1}D)\dot{x} + M^{-1}u \tag{4}$$

By substituting the input from (3) into (4) the final expression is

$$\ddot{x} = -M^{-1}(R + K_p)x - M^{-1}(D + K_d)\dot{x} + M^{-1}K_i z. \tag{5}$$

The equation is represented in state space with the standard $\dot{x}$ = Ax + Bu form.

$$\begin{bmatrix} \dot{\mathrm{z}} \\ \dot{\mathrm{x}} \\ \ddot{\mathrm{x}} \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{I} & 0 \\ 0 & 0 & \mathrm{I} \\ 0 & -\mathrm{M}^{-1}\mathrm{R} & -\mathrm{M}^{-1}\mathrm{D} \end{bmatrix} \begin{bmatrix} \mathrm{z} \\ \mathrm{x} \\ \dot{\mathrm{x}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\mathrm{M}^{-1} \end{bmatrix} [-\mathrm{K_i} \quad \mathrm{K_p} \quad \mathrm{K_d}] \begin{bmatrix} \mathrm{z} \\ \mathrm{x} \\ \dot{\mathrm{x}} \end{bmatrix} \tag{6}$$

2.1. Regional Eigenvalue Assignment for Performance

To control the system to meet certain time domain specifications Regional Eigenvalue Assignment is used. To reduce the amount of tuning required for a PID controller the time domain specifications are related to their 2nd order transfer function equivalent to find the corresponding region in the complex plane [8]. The three regions used are each bounded; with one being a shifted stable region bounded by α to bound the settling time; semicircle region bounded by ω_n to bound the natural frequency, and lastly a sector region bounded by θ to bound the damping ratio. The state matrix A is stabilizable within the region bounded by α, ω_n, and θ, which corresponds to decay rate, natural frequency, and damping ratio respectively if there is a solution to three LMI simultaneously which are found in [9]. To stabilize the system, the three LMIs are solved for LMI variables P and W to find the gain matrix K.

To add the additional constraints like H_2, the LMI variable P must be changed to Q where $Q = P^{-1}$ as the LMI for H_2 is in terms of Q. To solve multiple LMI simultaneously they must be written with the same LMI variable. Rederiving each individual LMI in terms of Q would be arduous especially for more complex regions and so a new subregion is more appropriate. The new inscribed subregion will carry over all the time domain characteristics of the larger region while remaining easier to modify.

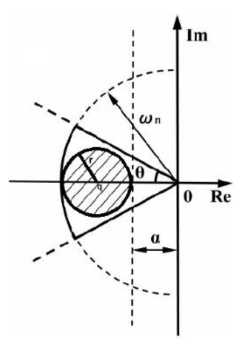

Fig. 1: Sub Region $S_{q.r}$

The region, shown in Fig 1 as the shaded circle, is the largest circle inscribed within the three regions defined whose radius is bounded by the natural frequency with the either decay rate or with the damping ratio; whichever bound is smaller. The first value possible for the radius of the new circle region uses the damping ratio and is called r_ζ. This radius is defined by turning angle θ into ζ where $\theta = cos^{-1}(\zeta)$. The radius of the largest circle would then be

$$r_\zeta = \frac{\omega_n sin(\theta)}{1 + sin(\theta)}. \tag{7}$$

The other radius, r_α is, bounded by the decay rate as is given by the following equation.

$$r_\alpha = \frac{\omega_n - \alpha}{2} \tag{8}$$

Once both possible radius values are found the value of r is chosen to be the smaller of the two radius values.

$$r = \begin{Bmatrix} r_\alpha, & if\ r_\zeta \geq r_\alpha \\ r_\zeta, & if\ r_\alpha > r_\zeta \end{Bmatrix} \tag{9}$$

From here the center of the circle region q is determined where $q = \omega_n - r$. To place the eigenvalues of the system within this region, a single LMI is solved for matrices P and W to determine the control gain K where $K = WP^{-1}$. The state feedback LMI for this region is derived in (10) and is very similar to the disk region LMI found in [9].

$$\begin{bmatrix} -rP & qP + AP + BW \\ qP + PA^T + W^T B^T & -rP \end{bmatrix} < 0 \tag{10}$$

2.2. H_2 Control with Regional Eigenvalue Assignment

The control of a system to achieve desired transient performance is only one step to obtain required behavior. Robustness criteria are a way to further improve the system performance. The performance criteria implemented in this work is H_2 control, as it would add optimality to the control with regards to a performance output. To meet this criterion a system must satisfy the following inequality [10],

$$\int_0^t ||Z||^2 \, d\tau \leq \frac{1}{\delta} \lambda_{max}(P) ||x(0)||^2 \tag{11}$$

where the H_2 gain, δ, is a positive scalar and Z is the performance output.

$$Z = C_Z x + D_Z u \tag{12}$$

By using the Lyapunov energy function,

$$V = x^T P x \tag{13}$$

the H_2 criteria found in [11], is given as

$$-\dot{V} - \delta Z^T Z > 0. \tag{14}$$

Next, the definition for the performance output is included in (14),

$$-(A + BK)^T P - P(A + BK) - \delta (C_Z + D_Z K)^T (C_Z + D_Z K) > 0. \tag{15}$$

Then by pre and post multiplying by Q, where $Q = P^{-1}$, and simplifying (15) the following LMI results,

$$-\mathrm{QA}^T - \mathrm{QK}^T\mathrm{B}^T - \mathrm{AP}^{-1} - \mathrm{BKQ} - \delta\mathrm{Q}(C_Z + D_Z K)^T(C_Z + D_Z K)\mathrm{Q} > 0. \tag{16}$$

The Schur's compliment lemma is used to separate the last term.

$$\begin{bmatrix} -Q\mathrm{A}^T - \mathrm{QK}^T\mathrm{B}^T - \mathrm{AQ} - \mathrm{BKQ} & \frac{\delta}{2}(C_Z + D_Z K)Q \\ \frac{\delta}{2}(C_Z + D_Z K)^T Q & I \end{bmatrix} > 0 \tag{17}$$

From here an auxiliary variable $W_q = KQ$ is used to make the inequality linear

$$\begin{bmatrix} -\mathrm{QA}^T - W_q^T\mathrm{B}^T - \mathrm{AQ} - \mathrm{BW}_q & \frac{\delta}{2}\left(\mathrm{QC}_Z^T + \mathrm{W}_q^T\mathrm{D}_Z^T\right) \\ \frac{\delta}{2}\left(C_Z Q + D_Z W_q\right)^T & I \end{bmatrix} > 0. \tag{18}$$

To combine H_2 control with the previous regional eigenvalue assignment we need to combine the two LMIs of (10) and (18) into one LMI with the same variables P and W. The LMI to place eigenvalues within a circle is linear in terms of P while the H_2 LMI is in terms of Q. To combine them we convert the circular region LMI variable to be solved for Q. To do this we first take the inequality that assures stability within a disk [12].

$$(\mathrm{A} + \mathrm{BK} + \mathrm{qI})^T P(\mathrm{A} + \mathrm{BK} + \mathrm{qI}) - r^2 P > 0 \tag{19}$$

Next, pre and post multiply by Q.

$$\mathrm{Q}(\mathrm{A} + \mathrm{BK} + \mathrm{qI})^T P(\mathrm{A} + \mathrm{BK} + \mathrm{qI})\mathrm{Q} - r^2 QPQ > 0 \tag{20}$$

Then using Schur's complement and substituting $W_q = KQ$ results in

$$\begin{bmatrix} -rQ & qQ + QA^T + QW_q^T \\ \mathrm{qQ} + \mathrm{AQ} + \mathrm{BW}_q & -rQ \end{bmatrix} > 0. \tag{21}$$

Lastly, combining both LMIs and solving for Q and W_q through the LMI solver in MATLAB the control gain K is solved where $K = W_q Q^{-1}$.

$$\begin{bmatrix} -rQ & qQ + QA^T + QW_q^T & 0 & 0 \\ \mathrm{qQ} + \mathrm{AQ} + \mathrm{BW}_q & -rQ & 0 & 0 \\ 0 & 0 & -\mathrm{QA}^T - \mathrm{W}_q\mathrm{B} - \mathrm{AQ} - \mathrm{BW}_q & \frac{\delta}{2}\left(QC_Z + W_q D_Z\right) \\ 0 & 0 & \frac{\delta}{2}\left(C_Z Q + D_Z W_q\right)^T & I \end{bmatrix} > 0 \tag{22}$$

The final LMI found in (22) allows for multiple criteria to be met. The first set of criteria to be met are the regional eigenvalue assignment criteria in the upper left hand two by two submatrix. Using the gain will assign the eigenvalues within the circle region defined by radius r and center q. The lower right hand two by two submatrix in (22) will assure H2 performance is met with regards to a performance output. By solving the LMI and finding a positive definite matrix Q will assure that all criteria are met. This control method can be done offline because the state matrices, performance criteria, and performance output parameters are known.

2.3. Reduced Order Observer

State feedback requires that values for all state variables be available to use to control the system. In situations where some of the state variables are not available, observers are used to estimate the whole or part of the state. A full order observer estimates all state variables while a reduced order observer (ROO) estimates only the unmeasured state variables. The reduced order observer often performs better than a full-order observer in a closed-loop control system as it does not redundantly estimate known state variables and requires fewer computations. The equation for a Luenburger observer of a closed loop system is shown next.

$$\dot{\hat{\mathrm{x}}} = \mathrm{A}\hat{\mathrm{x}} + \mathrm{Bu} + \mathrm{L}(\mathrm{y} - \mathrm{Cx}) \tag{23}$$

To design a ROO, the known and unknown state variables are partitioned in the following way,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mathrm{u} \tag{24}$$

where $x_1 \in R^{n_1}$ is measurable, $x_2 \in R^{n_2}$ is estimated, and $n_1 + n_2$ is the order of the system. If the system is not originally in this form, then a state transformation can be used to separate the measured and immeasurable states. Next the measured state equation is written, and it's assumed that $\dot{x}_1$ is available. This assumption is not necessary and will be lifted later.

$$\dot{\mathrm{x}}_1 = \mathrm{A}_{11}\mathrm{x}_1 + \mathrm{A}_{12}\mathrm{x}_2 + \mathrm{B}_1\mathrm{u} \tag{25}$$

Equation (25) allows for the construction of an observer that estimates $\hat{x}_2$ using (16) and after the equation is factored and like terms are collected the equation becomes

$$\dot{\hat{\mathrm{x}}}_2 = (\mathrm{A}_{22} - L\mathrm{A}_{22})\hat{\mathrm{x}}_2 + (\mathrm{A}_{21} - \mathrm{LA}_{11})\mathrm{x}_1 + (B_2 - \mathrm{L}B_1)\mathrm{u} + \mathrm{L}\dot{x}_1\,. \tag{26}$$

To eliminate $\dot{x}_1$an auxiliary variable is defined,

$$\mathrm{w} \triangleq \mathrm{x}_2 - \mathrm{Lx}_1 \tag{27}$$

and substituting the auxiliary variables estimate $\hat{\mathrm{w}} = \hat{\mathrm{x}}_2 - \mathrm{Lx}_1$ into (26) results in

$$\dot{\hat{w}} = \dot{\hat{\mathrm{x}}}_2 - \mathrm{L}\dot{\mathrm{x}}_1 = (\mathrm{A}_{22} - L\mathrm{A}_{12})\hat{\mathrm{w}} + (\mathrm{A}_{22}\mathrm{L} - \mathrm{LA}_{12}L + \mathrm{A}_{21} - \mathrm{LA}_{11})\mathrm{x}_1 + (B_2 - \mathrm{L}B_1)\mathrm{u}. \tag{28}$$

Finally, the convergence of the observer is determined by the eigenvalues of $A_{22} - LA_{12}$ where L is the observer gain. To use the estimated state variables, the observer must converge much quicker than the system converges. Because the designer picks the ideal eigenvalues of the observer, the observer gain L is found to meet the ideal case through eigenvalue placement. The conversion of the estimation of $\hat{w}$ into $\hat{x}_2$ is trivial. In our application, x is assumed to be available and $\dot{x}$ is estimated to avoid differentiation. So, the ROO is set up with the following variables.

$$\hat{x}_2 = \hat{w} + Lx_1 \tag{29}$$

$$A_{11} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & -\mathrm{M}^{-1}R \end{bmatrix}, \quad A_{22} = -\mathrm{M}^{-1}D, \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{30}$$

3. Simulations and Results

A mass connected to a rigid body by a spring and damper is commonly used as a standard test system for control designs. The differential equations of motion come from Newton's second law of motion. To test the proposed design method a commonly used standard test system, shown in Fig. 2, is used for simulation studies.

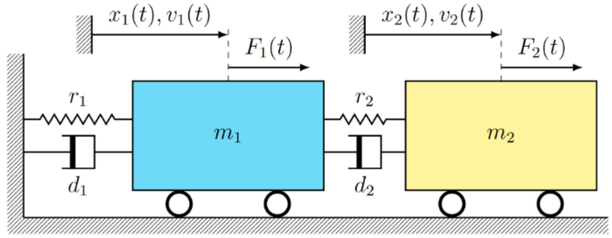

Fig. 2: Double Mass Spring Damper System

The mechanical system has two degrees of freedom. The two masses, m_1 and m_2, are connected by one spring and damper r_2 and d_2 respectivly while mass m_1 is connected to a rigid frame by one spring and damper r_1 and d_1. The external forces applied to the system are $F_1(t)$ and $F_2(t)$. For this system, the state space equations take the following form:

$$\dot{x}(t) = AX(t) + Bu(t) \tag{31}$$

$$y(t) = CX(t) + Du(t) \tag{32}$$

where $x(t)$ are the states and $u(t)$ is the input to the system. The state matrix is defined as follows where x is the relative position of the two masses and $\dot{x}$ is the velocity of each mass.

$$X = \begin{bmatrix} z_1 \\ z_2 \\ x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\dfrac{r_1 + r_2}{m_1} & \dfrac{r_1}{m_1} & -\dfrac{d_1 + d_2}{m_1} & \dfrac{d_2}{m_1} \\ 0 & 0 & \dfrac{r_2}{m_2} & -\dfrac{r_2}{m_2} & \dfrac{d_2}{m_2} & -\dfrac{d_2}{m_2} \end{bmatrix} \tag{33}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \dfrac{1}{m_1} & 0 \\ 0 & \dfrac{1}{m_2} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{34}$$

System parameters used for simulation are given in Table 1.

Table 1: Parameters for double mass spring damper system simulations

Parameter List					
Mass (kg)		Spring (N/m)		Damper (N*s/m)	
m_1	m_2	r_1	r_2	d_1	d_2
70	140	500	250	10	50

3.1. PID Control with Regional Eigenvalue Assignment

For this system the performance criteria goals of the systems closed loop response are defined to have a 1-meter steady state value of displacement, settling time under 1 second, rise time under 0.5 second, and a percent overshoot of under 10%. The following parameters are used. Damping ratio ζ = 0.59, natural frequency ω_n= 6.8, and decay rate α = 4.2 which are used as bounds to define the region used in the LMI. The largest circle that is inscribed within the region is found through Eq. (7-9). The radius found using natural frequency and damping ratio results in r_ζ = 3.0. The other radius bounded by the natural frequency and decay rate results in a radius of r_α = 1.5. The new region is the circle with the smaller of the two radii and the center of the circle is defined as q where $q = \omega_n - r = -5.3$; which results in a new circular region that has a center at q and a radius of r_α.

Next, the LMI found in (10) is solved to find a control gain matrix K which is used as PID gains in feedback with the system to obtain the step response of the system. To ensure the reduced order observer is estimating efficiently the derivative state is plotted alongside the estimate, where the estimate should converge quickly to the actual value. The plots of each are shown in Fig. 3 alongside the Simulink diagram showing the implementation of the controller.

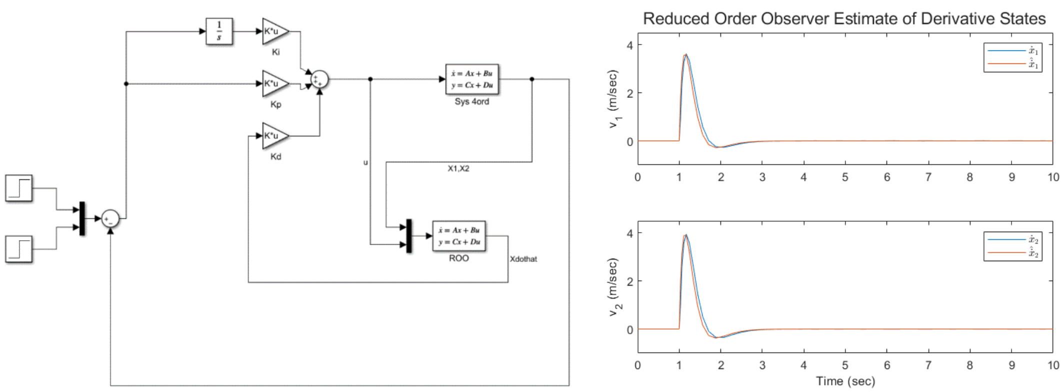

Fig. 3: Simulink implementation with simulated and estimated derivative states

By using these gains the rise time and settling time are met, however, percent overshoot is over the acceptable range. By increasing the magnitude of the derivative gains by 10% the final PID gains are given in (35). The step response is shown in Fig 4 together with the open loop step response. For the controlled system, the steady state error, response time, and the percent overshoot met specification.

$$K_P = -\begin{bmatrix} 8618 & 0 \\ 0 & 17235 \end{bmatrix}, \quad K_i = -\begin{bmatrix} 4452 & 250 \\ 250 & 10153 \end{bmatrix}, \quad K_d = -\begin{bmatrix} 786 & 50 \\ 50 & 2014 \end{bmatrix} \tag{35}$$

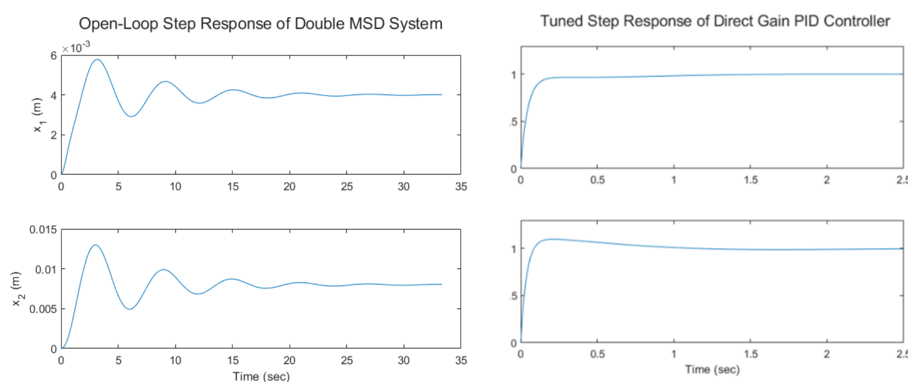

Fig 4: Step response of open loop and controlled system

The time domain performance criteria of the controlled system are found in Table 2.

Table 2: Step Response Performance Criteria of Controller System

x_1			x_2		
T_s	T_r	PO	T_s	T_r	PO
0.19 s	0.11 s	0%	0.75 s	0.07 s	9%

3.2. H_2 Control with Regional Eigenvalue Assignment

To add optimality to the regional eigenvalue assignment control, another constraint, the H_2 control criterion, is added. The LMI in (22) is solved to find control gains that when applied to the system will satisfy both H_2 and REA requirements. This control technique will use the same feedback input and the performance output variables C_Z and D_Z are defined as the following.

$$\mathrm{C}_z = 0.1 * I_6, \qquad \mathrm{D}_z = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \\ 0.1 & 0.1 \\ 0.1 & 0.1 \\ 0.1 & 0.1 \\ 01 & 0.1 \end{bmatrix} \tag{36}$$

Using the LMI solver in MATLAB, the unknown matrices Q and W_q are found which lead to the following PID gains.

$$\mathrm{K}_P = \begin{bmatrix} 10494 & 50804 \\ -12558 & -50177 \end{bmatrix}, \qquad K_i = \begin{bmatrix} 7067 & 31302 \\ -7952 & -31091 \end{bmatrix}, \qquad K_d = \begin{bmatrix} 852 & 4516 \\ -1165 & -5136 \end{bmatrix} \tag{37}$$

To meet H_2 control the optimal gain $1/\delta$ is minimized until there are no feasible solutions. For this system the optimal gain is $1/\delta$ = 0.00045. The eigenvalues of the control system are found to lie within the region and the performance output is found to satisfy the H_2 inequality found in (11). Simulation of the resulting step response of the system is shown in Fig. 6. The controlled system holds the H_2 performance criteria and still meets most of the time domain criteria defined. The new control method meets rise and settling times however it does not meet the percent overshoot only with regards to the second mass. The addition of the H_2 control criteria has added optimality to the control while keeping most of the time domain criteria designed for using REA.

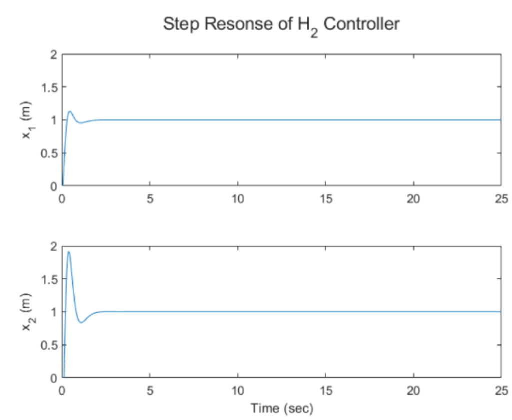

Fig. 5: Step response of the combination H_2 REA controller

4. Conclusion

The methodology developed in this work provides the ability to satisfy multiple criteria simultaneously using REA through LMIs. Solution of the LMIs is all done offline and provides a set of static PID gains to be used which requires a limited amount of additional tuning. The system simulated had two inputs and two outputs resulting in 12 gains to be tuned. The PID state feedback control resulted in eliminating the tuning. The controller gains used notably affected the performance criteria in different ways. Settling time and rise time were easier to satisfy when compared to percent overshoot. For the H_2 case similar results were observed however choosing the performance output differently or selecting tighter constraints could result in better performance. Differentiation of the state is also avoided using an ROO.

Further work in this area should include additional multivariable systems. This method can be easily applied to a linearized nonlinear electromechanical systems such as robot manipulators to show the usefulness technique. Taking this result into discrete time could also be useful and would require differences in the regions that correspond to the performance criteria. Lastly, the addition of even more performance criteria such as H_∞ and passivity conditions could accommodate system or measurement noise.

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