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Curve Tracking of Nonlinear Dynamic System Using Linear State-Space Model

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Abstract - In this paper, curve tracking of nonlinear dynamic systems is discussed. In mathematical modelling, a curve is defined as the solution of a dynamic system. Assuming the actual model of a dynamic system is unknown, we only have the solution curve of a system. Hence, tracking the curve becomes prominent in studying a nonlinear dynamic system. For this purpose, we propose a linear state-space model to track the curve of a nonlinear dynamic system. First, a least squares optimization problem is introduced, where the differences between the system and the linear model are defined. An adaptive parameter is introduced in the linear model, aiming to capture these differences. Second, the first-order necessary condition is derived, and the adaptive parameter is determined to update the curve of the linear model. Once convergence is achieved, the optimal solution curve of the linear model approximates the correct solution curve of the nonlinear system despite model-reality differences. Third, an example of a chemical kinetics system is studied for illustration. The simulation results show the efficiency of the computation algorithm, and the iterative solution demonstrates the accuracy of curve tracking. Therefore, using the linear state-space model to track the curve of the nonlinear dynamic system is satisfactorily handled.

Keywords**:** Curve Tracking, Nonlinear Dynamic System, Adaptive Parameter, Linear State-Space Model, Approximate Solution, Model-Reality Differences

1. Introduction

Nonlinear dynamic systems are commonly considered for the mathematical modelling of real-world problems [1]. The system solution can be employed to understand the essence of the problem and assist us in decision-making, especially for the solution curve of the system, which is relatively more preferred than the numerical solution. From the definition, the solution curve is referred to as the solution of a dynamic system presented in a graphical view [2]. Because of this, the system's behaviour and the system movement trend are observed clearly. Unfortunately, many real-world problems are revealed through data observation without knowing the mathematical models [3]. Their solution curves might not be smooth, and some solution curves display fluctuation behaviour, which could be natural phenomena or man-made mistakes.

In previous studies, Grenat et al. [4] proposed an efficient multi-parametric recursive continuation method for specific solution points of a nonlinear dynamical system. The topology of particular points found on the frequency response curves was explored by tracking extremum points in the successive codimensions of the problem concerning multiple system parameters. Liao et al. [5] proposed an optimization formulation integrated with the prediction-correction strategy to achieve bifurcation tracking. The bifurcation detection problem was formulated as a constrained optimization problem to locate the bifurcation periodic solution. Gonzalez et al. [6] developed an expression based on Fourier analysis for modelling and simulating nonlinear first-order systems when the explicit expressions of the characteristic curves are unknown. The result allows the system modelling and simulation based on the amplitude and phase Fourier spectrum using the Fast Fourier Transform. Anastasio et al. [7] studied the periodic solutions of nonlinear mechanical systems from the nonlinear state-space model estimated using the nonlinear subspace identification (NSI) technique. The nonlinear frequency response curves of the system were estimated to merge the harmonic balance method with the NSI technique.

In this paper, we discuss a linear state-space model [8] for tracking the solution curve of a dynamic system. For this purpose, a parameter is added to the model to measure the differences between the model and the system iteratively. To minimize these differences, a least squares optimization problem [9] is introduced. Then, the necessary optimality condition is derived, and a recursion equation is given to update the parameter's value. With this, the parameter adaptively adjusts the solution curve of the linear model to approximate the correct solution curve of the nonlinear system despite model-reality differences [10]. For illustration, a chemical reaction example is studied to demonstrate the efficiency of the computation algorithm proposed.

The rest of the paper is organized as follows. In Section 2, a general dynamic system is described. A linear state-space model is also described as a comparison. In Section 3, the methodology of using the linear state-space model for solving the nonlinear dynamic system is discussed. A least squares optimization is introduced, and the calculation procedure is summarized as an iterative algorithm. In Section 4, a chemical reaction example is studied for illustration. Finally, concluding remarks are made about the study conducted.

2. Problem Description

Consider a general dynamical system [11, 12],

$$\dot{x} = f(x),\ x(0) = x_0, \tag{1}$$

where $x \in \Re^n$ is an n-vector of state variables, and $f : \Re^n \times \Re \to \Re^n$ is the system dynamics. Here, $\dot{x}$ represents the rate of change of the state over time t and x_0 is the initial condition. The system (1) has a solution curve given by

$$x(t) = x_0 + \int_0^t f(x)\ dt, \tag{2}$$

and through discretization, the solution curve becomes

$$x(k+1) = x(k) + \Delta t_k f(x(k)), \tag{3}$$

where $\Delta t_k = t_{k+1} - t_k$ is the sampling time. Here, finding the solution curve (2) is complex because it involves solving an integral of a nonlinear function. Using the Riemann sum to the solution curve (3) can approximate the solution of the system (1), but the calculation procedure is costly [13] .

On the other hand, consider a linear state space model [8, 14],

$$\dot{x} = Ax,\ x(0) = x_0, \tag{4}$$

with $A \in \Re^{n \times n}$ is an $n \times n$ state transition matrix. The solution curve of the linear model (4) is expressed by

$$x(t) = x(0)e^{At}, \tag{5}$$

and its discrete solution is

$$x(t) = x(\tau)e^{A(t-\tau)}, \tag{6}$$

for $t > \tau$. It is noticed that solving the linear model (4) with the solution curve (5) is simple. However, solving the linear model (4) would not give the solution to the system (1), and the solution curve (5) does not approximate the solution curve (2).

Hence, we want to propose a computational algorithm that only solves the linear model iteratively. At the end of the calculation procedure, once convergence is achieved, the iterative solution can give a correct solution to the system (1) despite model-reality differences.

3. Computational Method

Define a least squares optimation problem [9],

$$\text{Minimize} \qquad J(\alpha) = (f(x) - Ax - \alpha)^{\mathrm{T}} (f(x) - Ax - \alpha), \tag{7}$$

subject to the linear state-space model

$$\dot{x} = Ax + \alpha, \; x(0) = x_0, \tag{8}$$

where $\alpha \in \Re^n$ is the adaptive parameter to be determined. Refer to the objective function (7), the first-order necessary condition is derived by

$$-2(f(x) - Ax - \alpha) = 0, \tag{9}$$

which leads to

$$\alpha = f(x) - Ax. \tag{10}$$

Thus, the adaptive parameter (10) measures the differences between the linear model and the system to minimize the objective function (7). In this situation, the linear model's solution curve closely tracks the nonlinear system's solution curve, and the prediction solution is satisfactorily presented by

$$\hat{x}(k+1) = \hat{x}(k) + h(A\hat{x}(k) + \alpha(k)), \; \hat{x}(0) = x_0, \tag{11}$$

with h is the appropriate step size.

4. Illustrative Example

Consider the chemical reaction [15, 16], which is also known as the Belousov-Zhabotinsky (BZ) reaction and the Oregonator model, given by

$$\frac{dx_1}{dt} = \alpha(x_2 - x_1 x_2 + x_1 - \beta x_1^2), \tag{12}$$

$$\frac{dx_2}{dt} = \frac{1}{\alpha}(x_3 - x_1 x_2 - x_2), \tag{13}$$

$$\frac{dx_3}{dt} = \gamma(x_1 - x_3), \tag{14}$$

where x_1 is the concentration of bromous acid ($HBrO_2$), x_2 is the concentration of bromide ions (Br^-), x_3 is the concentration of cerium oxides (Ce(IV)), and α, β, γ are kinetic constants. Here, we uses the initial conditions $x_1(0) = 4$, $x_2(0) = 1.1$ and $x_3(0) = 4.4$, while the values of kinetic constants are $\alpha = 77.27$, $\beta = 8.375\times10^{-6}$ and $\gamma = 0.161$, respectively. Figure 1 shows the solution curve of the chemical reaction. It can be seen that the concentration of cerium oxides (x_3) expresses a jump to the value of 22,041 at the beginning time of 6.06 seconds and another jump to the value of 21,438 at 309.09 seconds. The concentrations of bromous acid (x_1) and bromide ions (x_2) are relatively small compared with the concentration of cerium oxides (x_3).

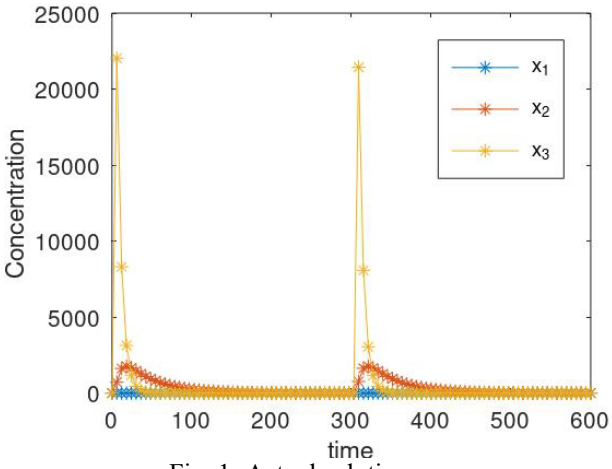

Fig. 1: Actual solution curve.

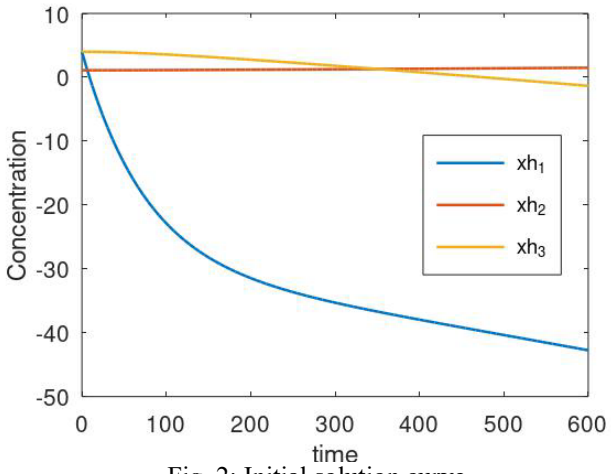

Fig. 2: Initial solution curve.

Consider a linear state-space model that we introduced,

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix}-7.7322 & -231.8100 & 0\\ -0.0142 & -0.0647 & 0.0129\\ 0.1610 & 0 & -0.1610\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} + \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}, \tag{15}$$

with $\alpha = (\alpha_1, \alpha_2, \alpha_3)^{\mathrm{T}}$ is the adaptive parameter vector. The solution curve of the model is shown in Figure 2, which the concentration of bromous acid (x_1) with negative values is not a realistic solution curve. We assign this solution as the initial solution before applying the computational algorithm proposed, and the adaptive parameter α is a zero vector.

Since the concentration of cerium oxides (x_3) shows an extremely large value for the sudden jumps, we convert the solution curve into the natural logarithm for better visualization. Figure 3 shows the solution curve of the model (15), where the adaptive parameter α captures the differences between the model and the system. The solution curve presents the oscillation in the cycle time, and it is noticed that the predicted solution closely tracks the system's actual solution curve. The prediction errors are shown in Figure 4 and express the oscillation trend.

In addition, Figure 5 shows the solution curve of the adaptive parameter α at the end of the calculation procedure. These differences between the model and the system exist and assist the model's solution curve to track the system's solution curve satisfactorily. The loss function, which represents the operating cost of using the linear state-space model, is shown in Figure 6 with a minimum value of 2.6838 in its natural logarithm.

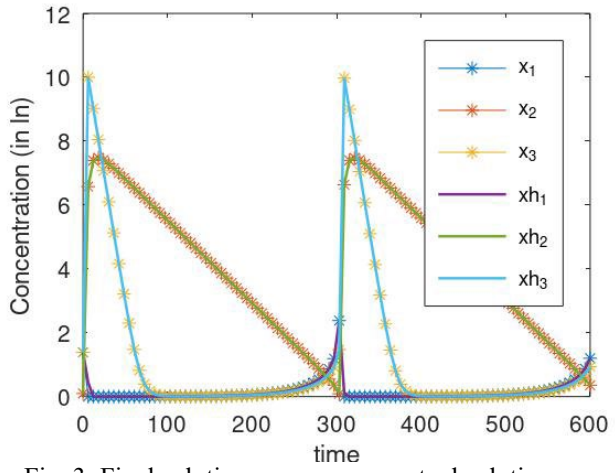

Fig. 3: Final solution curve versus actual solution curve (in natural logarithm, ln).

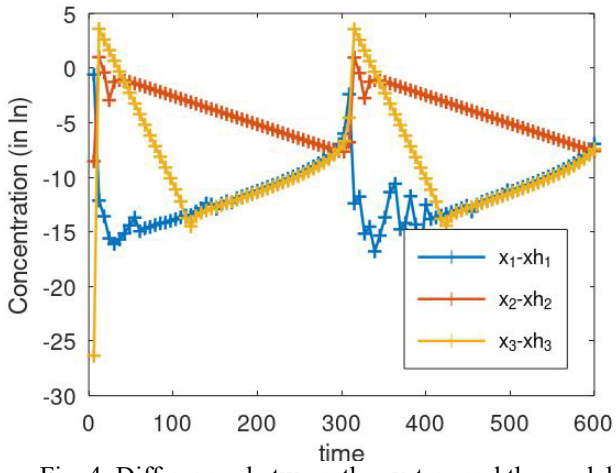

Fig. 4: Differences between the system and the model (in natural logarithm, ln).

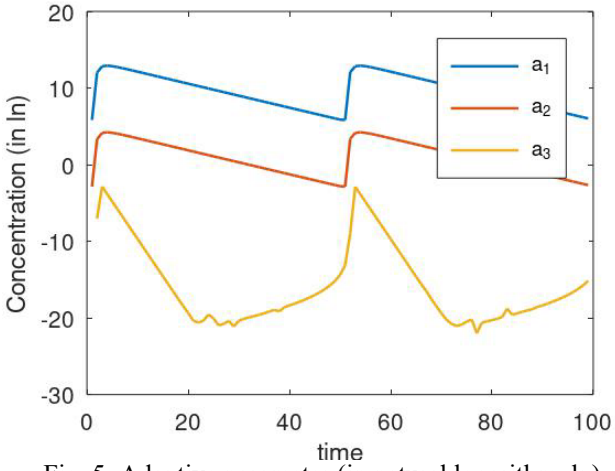

Fig. 5: Adaptive parameter (in natural logarithm, ln).

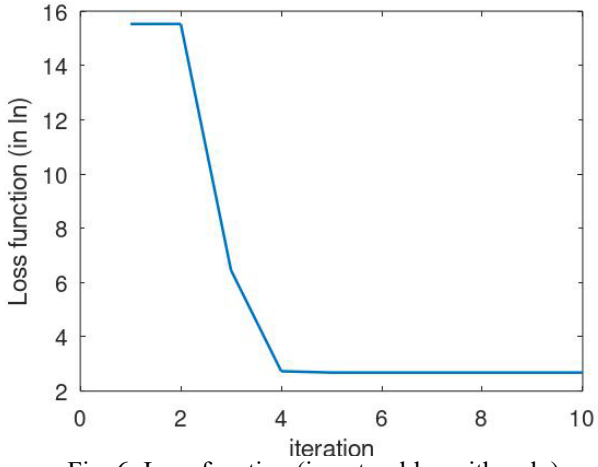

Fig. 6: Loss function (in natural logarithm, ln).

Table 1 shows the simulation result of the computational algorithm to obtain the prediction solution. The computational algorithm takes nine iteration numbers with an elapsed time of 0.8391 seconds to converge to the correct solution of the system within a tolerance 10^{-10}. From this result, the initial loss function has a large value of 5.5344106, and the final value of the loss function is 14.641. Therefore, the accuracy of the computational algorithm is demonstrated.

Table 1: Simulation result for computation algorithm.

Iteration Number	Elapsed Time (second)	Initial Loss Function	Final Loss Function
9	0.8391	5.5344×10^{6}	14.641

5. Concluding Remarks

The paper discussed using a linear state-space model to track the solution curve of a nonlinear dynamic system. The special feature of the study is adding a parameter in the model to measure the differences between the model and the system adaptively. By solving a least squares optimization problem, the adaptive parameter is generated. With this, the model's

solution curve approximates the system's correct solution curve despite model-reality differences. The example of a chemical reaction was studied for illustration. The simulation results demonstrated the accuracy of the prediction solution. Hence, the performance of the proposed computational algorithm was verified. For future research, tracking the nonlinear dynamic system in a random environment, including raw data observation, is recommended.

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