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Cracked Transformed Moment of Inertia of Steel-Reinforced and FRP-Reinforced Rectangular Beams and Slabs

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Abstract – Cracking in reinforced concrete (RC) slabs and beams subjected to flexural moments reduces their flexural stiffness significantly. At service load conditions, deflections are typically calculated using the effective moment of inertia (I_e), which is calculated based on the gross moment of inertia (I_g) and the cracked moment of inertia (I_{cr}). The latter is based on the transformed properties of the cross-section and requires the calculation of the depth of the compression zone (kd) then the calculation of the moment of inertia. This paper shows that (I_{cr}) depends solely on the modular reinforcement ratio ($\rho\, n$) where (ρ) is the reinforcement ratio and (n) is the modular ratio of the reinforcement relative to the concrete. It has been proposed in a previous study that ($\rho\, n$) ranges from 0.005 to 0.16. This range covers the properties of beams and slabs with concrete compressive strength (f_c') ranging from 20 to 60 MPa and with reinforcement ratio ranging from minimum to maximum values allowed by the ACI 318 building code. It also covers a wide range of beams and slabs reinforced with FRP bars. Within such a range of ($\rho\, n$), it will be shown that (I_{cr}) can be calculated directly using ($I_{cr} = 0.36\, (\rho\, n)^{0.81}\, b\, d^3$), where ($b$) and ($d$) are the width and effective depth of the cross-section respectively. The error between the exact and the approximate equation is limited. Based on this approximate equation, it is shown that the flexural stiffness of fully cracked elements depends on the ratio and the modulus of elasticity of the reinforcement raised to the power 0.8 and to (d) raised to the power 2.2. In addition, it is shown that the effect of (f_c') on the flexural stiffness is limited.

***Keywords*:** concrete, cracking, deflections, flexural stiffness, moment of inertia.

1. Introduction

Before flexural cracking, deflections in RC beams and slabs can be calculated based on the uncracked stiffness of the cross-sections. The effect of the reinforcement is typically neglected, and the flexural stiffness can hence be taken as $(E_c\, I_g)$, where (E_c) is the modulus of elasticity of concrete in compression [1]. Figure 1 shows a singly reinforced rectangular cross-section. The gross moment of inertia can is calculated as $(I_g = b\, h^3/12)$ where (h) is the full height of the cross-section.

After cracking, the flexural stiffness drops significantly. Building codes such as the ACI-318 building code [1] allows the use of the effective stiffness (I_e) using the following equations:

$$for \quad M_a \le \frac{2}{3}\, M_{cr} \qquad\qquad I_e = I_g \tag{1-a}$$

$$for \quad M_a > \frac{2}{3}\, M_{cr} \qquad\qquad I_e = \frac{I_{cr}}{1 - \left(\frac{2/3\, M_{cr}}{M_a}\right)^2 \left(1 - \frac{I_{cr}}{I_g}\right)} \tag{1-b}$$

In Eq. (1), the term (M_a) is the maximum applied bending moment in the member and (M_{cr}) is the flexural cracking moment.

Figure 1 shows the cracked cross section after transforming it to an equivalent concrete cross-section using the stiffness modular ratio (n) given by (E_r/E_c). The term (E_r) is the modulus of elasticity of the reinforcing bars. The modulus of elasticity of the concrete can be calculated as [1]:

$$E_c = 4700\sqrt{f_c'} \quad \text{in MPa} \tag{2}$$

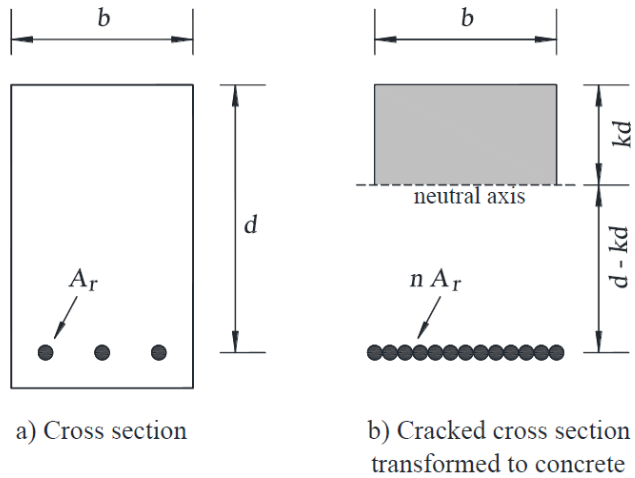

Fig. 1: Gross and cracked cross-sections of singly reinforced flexural element.

In Figure 1, the depth of the compression zone is (kd) where (k) is a unitless fraction smaller than unity. The term (k) can be calculated as:

$$k = \sqrt{2(\rho\, n) + (\rho\, n)^2} - (\rho\, n) \tag{3}$$

The cracked moment of inertia is calculated as:

$$I_{cr} = {}^{b\,(kd)^3}\!/_{3} + n\, A_r (d - kd)^2 \tag{4}$$

where the term (A_r) is the area of reinforcement in the tension zone. See Fig. 1. Replacing the term (A_r) in Eq. (4) with $(\rho \times b \times d)$ and the term (k) with the right-hand side of Eq. (3) and re-arranging gives:

$$I_{cr} = \left[\frac{1}{3}\left(\sqrt{2(\rho n) + (\rho n)^2} - (\rho\, n)\right)^3 + (\rho n)\left(1 - \left(\sqrt{2(\rho n) + (\rho n)^2} - (\rho n)\right)\right)^2\right] bd^3 \tag{5}$$

Equation (5) can be re-written as:

$$I_{cr} = [\eta]\ bd^3 \tag{6}$$

where the term (η) is given by:

$$\eta = \frac{1}{3}\left(\sqrt{2(\rho n) + (\rho n)^2} - (\rho\, n)\right)^3 + (\rho n)\left(1 - \left(\sqrt{2(\rho n) + (\rho n)^2} - (\rho n)\right)\right)^2 \tag{7}$$

Eq. (7) shows that the term (η) can be solely calculated using the term $(\rho \times n)$ without the need to calculate the term (k). Figure 2 plots the relationship between (η) and $(\rho \times n)$.

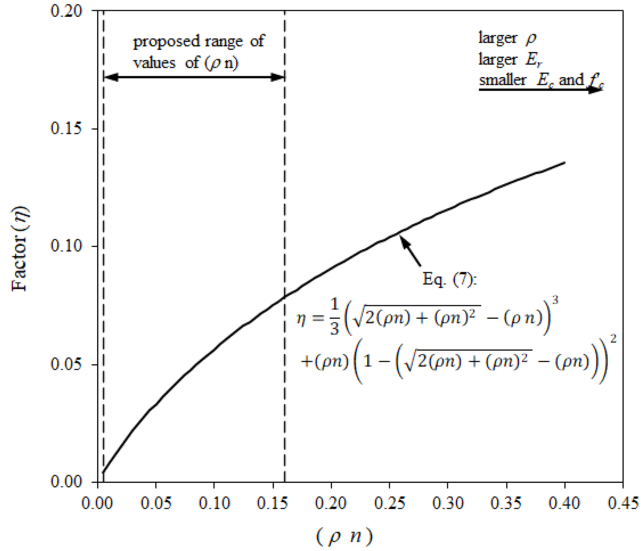

Fig. 2: Plot of (η) versus $(\rho \times n)$.

This paper shows that it is possible to develop a simple and direct relation between (η) and $(\rho \times n)$ if a practical range of the latter term is obtained. However, this is possible if the range of values of $(\rho \times n)$ is specified and selected based on practical design considerations.

2. Selected range of $(\rho \times n)$

The ACI 318 code [1] requires that steel-reinforced flexural members are designed to be tension-controlled. This is achieved by ensuring that at ultimate conditions, the largest strain in the tensile reinforcement exceeds the yield strain (ε_y) of this steel by at least 0.003. The code also requires that the amount reinforcement is not smaller than a certain minimum. The equations for minimum reinforcement are different for slabs and for beams.

For RC members with concrete compressive strength (f'_c) ranging from 20 to 60 MPa and reinforcing steel of yield strength equal to 420 MPa or 550 MPa, it is possible to prove that reinforcement between the minimum to the maximum levels leads to $(\rho \times n)$ ranging from 0.084 to 0.161 [2].

For FRP-reinforced members, the properties of the FRP bars vary considerably depending on the type of fiber in the bars and even with bars of similar type of fibers. To obtain guidance to a practical range of modular reinforcement ratio $(\rho \times n)$, the author developed a database of test results from beams and slabs reinforced with FRP bars from the literature [2]. The data was collected from thirty studies that were published between 1995 and 2019. Data from 233 beams and slabs longitudinally reinforced with FRP bars was available. he modular reinforcement ratio $(\rho \times n)$ ranged from 0.0047 to 0.116, with an average of 0.0215.

Based on the above it is proposed that practical values of $(\rho \times n)$ range from 0.005 to 0.16. It is to be noted that this range is suitable for steel-reinforced members with (f'_c) larger than 60 MPa, except that it is suitable within a partial rather than the full range of reinforcement levels (from minimum to maximum ρ).

3. Proposed Equation for (η)

This section presents the proposed equation and highlights how it can be used to assess the effects of different parameters on the moment of inertia of cracked transformed cross-sections reinforced with steel or with FRP bars.

3.1. Proposed Equation

It is proposed that for the range of $(0.005 \leq \rho \times n \leq 0.16)$, it is possible to calculate (η) directly from the following equation:

$$\eta = 0.36\,(\rho \times n)^{0.81} \tag{8}$$

and consequently, the term (I_{cr}) using the following equation:

$$I_{cr} = \; 0.36\,(\rho \times n)^{0.81} \;\; bd^3 \tag{9}$$

The use of Eq. (9) avoids the need to calculate (k) using Eq. (3) then (I_{cr}) using Eq. (4). It also provides a direct relation between (I_{cr}) and $(\rho \times n)$.

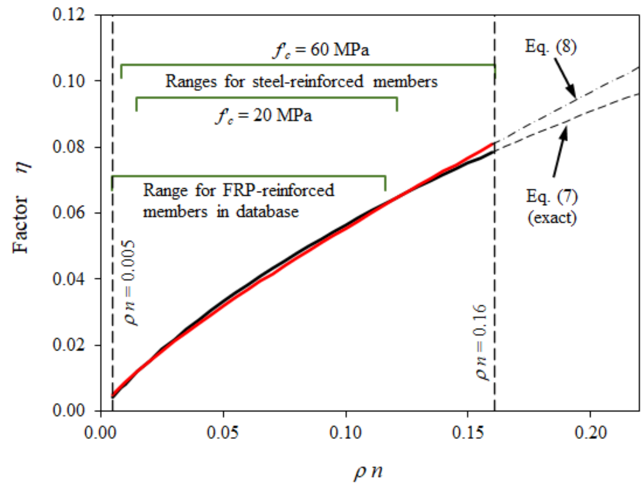

Fig. 3: Comparison between results of exact and proposed equations of (η).

Fig. 3 compares the exact values of (η) calculated using Eq. (7) to the approximate ones calculated using the proposed Eq. (8). If the selected range is split into fifty equidistant points and the ratio of the exact to the calculated values of (η) are calculated, the average of the ratios is 1.01 and the coefficient of variation is 3.2%. The ratios range from 0.89 to 1.04. For the 233 members in the database, the ratio of exact to approximate (η) ranged from 0.88 to 1.04. The average of the 233 ratios was 1.00 and the coefficient of variation was 3.9%.

3.2. Application

Building codes such as the ACI 318 code [1] allows the deflections of steel-reinforced flexural members to be calculated based on the effective moment of inertia (I_e) such as the one given in Eq. (1-b). The ACI draft code for the design of flexural members reinforced with glass FRP (GFRP) bars [3] allow a similar approach. The value of (I_e) ranges from an upper limit equal to (I_g) to a lower limit equal to (I_{cr}). In practical applications, the value of (I_e) approaches the lower limit and designers conservatively assume the lower limit of (I_{cr}) in their preliminary calculations. On the other hand, the Canadian standard for the design and construction of building components with FRP [4] assumes that the moment of inertia in cracked regions of FRP-reinforced members is equal to (I_{cr}).

The maximum deflection (Δ) in a simply supported flexural member of span (ℓ) subjected to a uniformly distributed load of magnitude (w) is calculated using the following equation:

$$\Delta = \frac{5}{384}\,\frac{w\,\ell^4}{(E_c I_e)} \tag{9}$$

The maximum deflection is inversely proportional to the flexural stiffness $(E_c\, I_e)$. In continuous beams, the maximum maximum deflection is also inversely proportional to $(E_c\, I_e)$. If $(E_c\, I_e)$ is conservatively taken as $(E_c\, I_{cr})$, it is of interest to to expand this flexural stiffness as follows:

$$E_c\, I_{cr} \approx E_c\ \times 0.36\,(\rho\, n)^{0.8}\ \, b\, d^3 \tag{10}$$

The power of the modular ratio in Eq. (10) is changed from 0.81 to 0.8 for simplicity. If the reinforcement ratio is replaced with $(A_r/b/d)$ and the modular ratio (n) is replaced with the ratio of the materials moduli $(E_r\ /\ E_c)$ and the equation is re-arranged, the following is obtained:

$$E_c\, I_{cr} \approx 0.36 \times E_c^{0.2} \times (A_r\ \times E_r)^{0.8}\ \times b^{0.2} \times d^{2.2} \tag{11}$$

Eq. (11) shows that the factor which has the largest effect on the flexural stiffness of well-cracked elements is the effective depth (d) followed by the area and the modulus of elasticity of the reinforcing bars. It is of interest to note that the flexural stiffness is proportional to (d) raised to the power 2.2 and not 3 as would be expected. It is also proportional to $(A_r\, E_r)$ raised to the power 0.8. This gives a direct assessment of the effect of the lower modulus of elasticity of FRP bars relative to steel bars on the stiffness and deflections of beams and slabs. Eq. (11) also shows that the modulus of elasticity of the concrete has a limited effect on the flexural stiffness. With the modulus of elasticity of concrete being proportional to the square root of (f_c') as shown in Eq. (2), it is shown that the flexural stiffness is proportional to $(f_c')^{0.1}$. This shows that the compressive strength of concrete has a negligible effect on the maximum deflections of well cracked flexural members.

4. Conclusion

This paper has shown that for rectangular concrete sections reinforced with steel of FRP bars, it is possible to calculate the cracked transformed moment of inertia (I_{cr}) directly from the modular reinforcement ratio $(\rho \times n)$. The proposed equation $I_{cr} =\ 0.36\,(\rho \times n)^{0.81}\ \ bd^3$ was shown to give relative accurate results for values of $(\rho \times n)$ ranging from 0.005 to 0.16. These values cover a wide range of possible design values in practice.

The paper also showed that the flexural cracked stiffness is proportional to the effective depth of the cross section raised to the power 2.2 and the area and modulus of elasticity of the reinforcement, both raised to the power 0.8. The effect of the compressive strength on the flexural stiffness is shown to be negligible.

Acknowledgements

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