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Shape Optimisation of Structural Elements in Flexure for Deflection Control

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Abstract **–** This paper presents an analytical method for optimising the shape of statically determinate structural elements in flexure aiming at minimising the maximum deflection. This analytical method provides an efficient evaluation of the optimal section to add material needed for reducing the deflection. The detailed mathematical derivation of the analytical method is provided, including symmetric and unsymmetric cases. To evaluate the performance of the proposed method, an optimisation example is presented, which shows the proposed method has better performance than the existing method [1]. The applicability of the proposed method is discussed at the end of the paper.

Keywords**:** shape optimisation; deflection; structural elements; analytical method.

1. Introduction

Structural optimisation includes mainly three categories: size optimisation, shape optimisation and topology optimisation [2]. Among these three types, shape optimisation focuses on finding the optimal shape of structural elements, aiming at minimising the defined objective function (such as mass) whilst satisfying imposed constraints (such as displacements, stresses and natural frequencies). In the domain of civil engineering, shape optimisation has been applied in the design of many types of structures, from beams, dams, to shell structures [3]–[6]. By shape optimisation, the volume of material can be reduced, leading to less CO_2 emissions.

Shape-optimised concrete structural elements have variable sections, which cannot be realised by traditional prismatic formworks but other formwork techniques, such as fabric formwork [7]. To really use them in concrete structures, these shape-optimised structural elements need to satisfy the conditions of both Ultimate Limit State (ULS) and Serviceability Limit State (SLS).

For optimising structural beams in reinforced concrete, previous studies mainly focus on the shape optimisation for ULS, ensuring that each section has adequate flexural and shear strength resisting to the corresponding bending moment and shear force, whilst shape optimisation for SLS has been inadequately studied. However, shape-optimised beams are more prone to the problem of excessive deflection than their prismatic counterparts due to the reduction of flexural stiffness [1]. The shape optimisation for deflection control is therefore worth investigating.

Previous work in SLS optimisation [1] made simplifying assumptions that may lead to inaccurate results in statically determinate beams. This paper improves on such previous approaches by returning to first principles to propose a revised analytical method which allows the optimisation of statically determinate structural elements in flexure for deflection control. The mathematical derivation will be given in detail in this paper.

2. Analytical Method

2.1. Hypotheses

The objective of the shape optimisation problem discussed here is to reduce the maximum deflection of a beam so that it does not exceed the maximum allowable deflection at SLS. The original shape can be either prismatic or variable-section with the shape obtained from ULS-based shape optimisation. As the starting point of the optimisation, it is assumed that, with the original shape, the maximum deflection cannot meet the SLS requirement, so that the shape optimisation for deflection control is necessary. Then, the question is how to increase the flexural stiffness of the beam by adding materials somewhere along the span. The optimisation problem is therefore transformed into: finding the position of the best section along the span where adding a unit mass of material can lead to the maximum decrease in the overall maximum deflection of the beam.

A simply-supported beam can be taken as example, as shown in Fig. 1. The section is rectangular and variable along the span. This original shape is assumed to be obtained from ULS-based design – every section is designed to be capable of resisting to the bending moment and shear force, however, the maximum deflection, which appears at the mid-span section, exceeds the maximum allowable value. It is therefore necessary to determine where to add material in order to reduce the maximum deflection in the most efficient way.

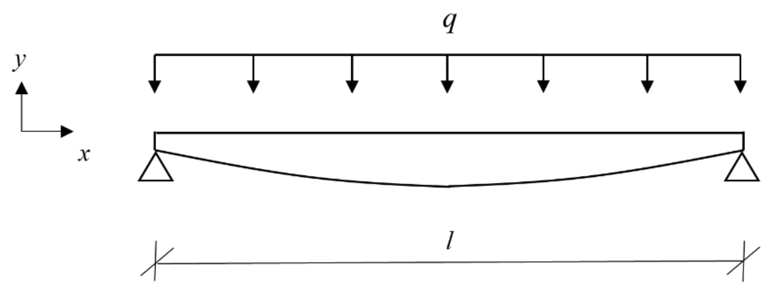


Fig. 1: Shape optimisation of a simply supported beam under uniformly-distributed loads for deflection control

It is hypothesised that the beam is Euler–Bernoulli beam and the shear effect can be ignored:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\mathrm{E}I(x)\frac{\mathrm{d}^2\Delta}{\mathrm{d}x^2}\right) = q(x) \tag{1}$$

where Δ is the deflection, q is the load, E is the elastic modulus (constant) and I is the moment of inertia of the section.

2.2. $\partial\Delta/\partial I_i$ in symmetric cases

In general, the deflection of a beam at $x = x_i$ can be obtained by integration from where the boundary condition is applied ($x = 0$) to the section at $x = x_i$, as shown in Eqs. (2)-(3) below, where θ is the rotation, κ is the curvature, θ_0, Δ_0, θ_i and Δ_i are respectively the rotation and the deflection at $x = 0$ and at $x = x_i$.

$$\theta_i = \int_{x=0}^{x=x_i} \kappa dx + \theta_0 \tag{2}$$

$$\Delta_i = \int_{x=0}^{x=x_i} \theta dx + \Delta_0 \tag{3}$$

The value of curvature κ_i of each section can be obtained by Eq. (4):

$$\kappa_i = \frac{\mathrm{d}^2\Delta}{\mathrm{d}x^2} = -\frac{M_i}{\mathrm{E}I_i} \tag{4}$$

where M_i and I_i are respectively the bending moment and the moment of inertia of the section at $x = x_i$.

For the beam in Fig. 1, it is obvious that the maximum deflection appears at the mid-span where $x = L/2$. Therefore, one important step is to evaluate the derivative $\partial\Delta/\partial I_i$ of each section. The complexity of this is that the rotation θ_0 at the support is not constant – its value depends on the rigidity of all sections along the span.

One solution is to assume the value of rotation at the support θ_0 and obtain the rotation of all sections along the span according to Eq. (2), then calculate the deflection according to Eq. (3) and check whether the deflection at another support meets the given boundary condition (here the deflection at the right support Δ_{2n} is 0). If this boundary condition cannot be met, the initial assumed value of θ_0 can be adjusted till the correct value can be found after several iterations. However, this process often needs computers, and a straightforward analytical solution for finding the most efficient place to add material cannot be easily obtained.

In order to solve this problem, it could be useful to take advantage of the symmetry of the defined problem. Owing to the symmetry of the beam and of the external loads, the rotation at the mid-span is zero. Therefore, the integration by Eq. (2) can start from the mid-span where the rotation is known as zero.

As shown in Fig. 2, the whole span can be divided into 2n segments, and d is the length of each segment. The deflection of every section can be obtained by Eq. (5) in discrete form:

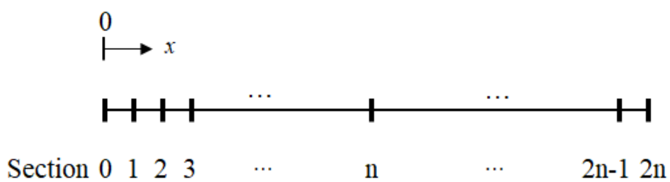


Fig. 2: Discretisation of the beam

$$\begin{aligned}
&\theta_n = 0\\
&\theta_{n-1} = \kappa_n d\\
&\theta_{n-2} = \kappa_n d + \kappa_{n-1} d\\
&\ldots\\
&\theta_1 = \kappa_n d + \kappa_{n-1} d + \cdots + \kappa_2 d\\
&\theta_0 = \kappa_n d + \kappa_{n-1} d + \cdots + \kappa_2 d + \kappa_1 d
\end{aligned} \tag{5}$$

where θ_0, $\theta_1 \ldots \theta_n$ and κ_0, $\kappa_1 \ldots \kappa_n$ are the rotations and the curvatures at the sections $x = x_0, x_1 \ldots x_n$, x_0 is the section at the left support, and x_n is the mid-span section.

The deflection of every section can be obtained by Eq. (6) in discrete form:

$$\begin{aligned}
&\Delta_0 = 0\\
&\Delta_1 = \Delta_0 + \theta_0 d\\
&\Delta_2 = \Delta_0 + \theta_0 d + \theta_1 d\\
&\ldots\\
&\Delta_n = \Delta_0 + \theta_0 d + \theta_1 d + \cdots + \theta_{n-1} d
\end{aligned} \tag{6}$$

Combining Eqs. (5)-(6), it can be obtained that:

$$\Delta_n = d \sum_{i=1}^{n} \kappa_i i d \tag{7}$$

Combining Eqs. (4) and (7), it can be obtained that:

$$\Delta_n = d \sum_{i=1}^{n} \frac{M_i}{\mathrm{E} I_i} \left(\frac{L}{2} - l_i \right) \tag{8}$$

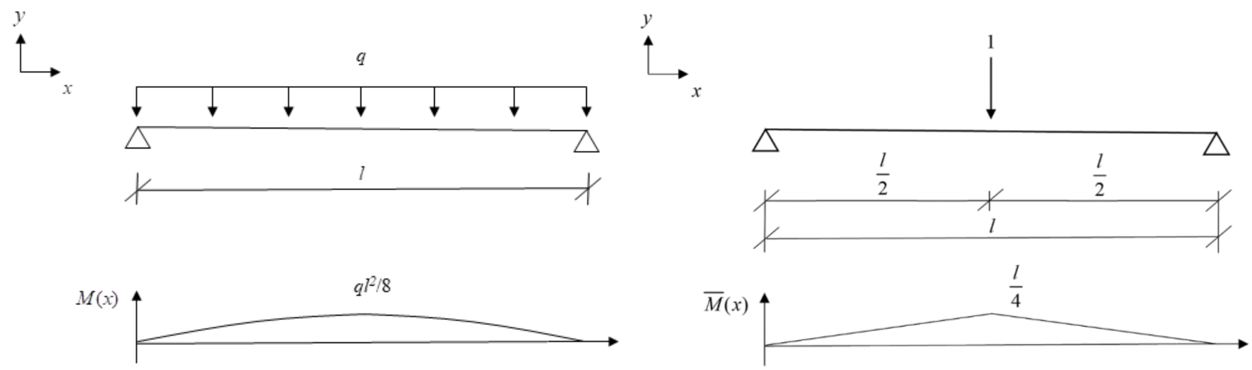
where l_i is the distance between the section i and the mid-span section.

When reducing the mid-span deflection by adding material to a certain section, the change in the bending moment to the slight change of self-weight can be neglected. The bending moment due to external loads will not change neither when there is any change in the variable section since the studied beam is statically determinate. Therefore, the M_i can assumed to be constant when any change happens in I_i.

Then, it can be seen from Eq. (8) that a slight change in the moment of inertia of the section i will lead to the change in the mid-span deflection, which can be written in the continuous form when $d \to 0$:

$$\frac{\partial \Delta_n}{\partial I_i} = -\frac{M_i}{\mathrm{E}{I_i}^2}\left(\frac{L}{2} - l_i\right) \tag{9}$$

2.3. $\partial\Delta/\partial I_i$ in unsymmetric cases



(a) Diagram of M (b) Diagram of $\overline{M}$

Fig. 3: Diagrams of bending moments (M: the bending moment under real loads; $\overline{M}$: the bending moment under virtual unit load)

The limitation of the mathematical derivation above is that the beam and the external loads should be symmetric. In order to extend Eq. (9) to more general cases, the method of virtual power can be used.

Assuming there is a unit load in the mid-span, the diagram of the distribution of bending moment $\overline{M}$ can be drawn, as shown in Fig. 3a. The diagram of the distribution of bending moment M under real loads can also be drawn, as shown in Fig. 3b.

The mid-span deflection can be obtained by the integration:

$$\Delta_n = \int \frac{M\overline{M}}{\mathrm{E}I} dl \tag{10}$$

$$\frac{\partial \Delta_n}{\partial I_i} = -\frac{M\overline{M}}{E{I_i}^2} \tag{11}$$

It can be seen from Fig. 3b that:

$$\overline{M} = \frac{1}{2}\left(\frac{L}{2} - l_i\right) \tag{12}$$

Combing Eqs. (10)-(12), it can be obtained that:

$$\frac{\partial \Delta_n}{\partial I_i} = -\frac{M_i}{2\mathrm{E}{I_i}^2}\left(\frac{L}{2} - l_i\right) \tag{13}$$

Comparing Eqs. (9) and (13), it can be observed that the right-hand side of Eq. (9) is twice of that of Eq. (13). In fact, fact, owing to the hypothesis of symmetry based on which Eq. (9) is derived, any change in I_i is assumed to happen for both two symmetric sections at the left and right sides of the beam. In more general cases when the beam and loads are not necessarily symmetric, if the change in I_i only happen for one single section, it will lead to the change in the mid-span deflection, of which the expression is given by Eq. (13).

2.4. $\partial I_i/\partial h$

Eq. (13) provides important information about where the increase of I can help reduce the most Δ_n. However, taking directly the value of derivative $\partial\Delta/\partial I_i$ by Eq. (13) to determine which section to add material will lead to the wrong answer, because there is still a gap between the increase of I and the addition of a unit mass of material, of which the relation is not necessarily linear.

The beam in Fig.1 with variable rectangular sections can be taken as an example. A unit mass of material can be added to increase the depth or the width of a certain section along the span. Since increasing the width is much less efficient than increasing the depth when the aim is to increase the I of a section, it is therefore assumed that the unit mass of material will be added to increase only the depth of section h by δh, and the width of the section b is constant along the span. Then it can be obtained that:

$$\frac{\partial I_i}{\partial h_i} = \frac{b{h_i}^2}{4} \tag{14}$$

It can be seen from Eq. (14) that an incremental increase of the section depth δh at different position along the span can lead to different increase of I. With regards to $\partial I_i/\partial h$, the most efficient position to increase the depth of the section whose depth is already the largest. This is intuitive because when the original depth of the section is larger, the added unit mass of material will be located further from the neutral axis of the section, leading to more significant increase of I.

For other shapes of section (e.g., circular sections, pipe sections, I sections, etc.), the relation similar to Eq. (14) can also be obtained mathematically.

With regards to cracked sections in two materials, such as steel-reinforced concrete, the moment of inertia of cracked sections can be used:

$$\frac{\partial I_i}{\partial h_i} = \frac{\partial I_e}{\partial h_i} \tag{15}$$

where I_e is the effective moment of inertia, whose value falls between the moment of inertia of gross section I_g (uncracked) and the moment of inertia of cracked section I_{cr} (fully cracked), which can be calculated following the method of ACI 318-19 [8] for sections in steel-reinforced concrete.

2.5. Final expression of $\partial\Delta/\partial m$

Assuming a unit mass of material can be added to a certain section along the beam in Fig. (1), it can be obtained:

$$\frac{\partial h}{\partial m} = \frac{1}{b\rho} \tag{16}$$

where m is the mass per unit length, ρ is the density.

Finally, the expression of $\partial\Delta/\partial m$ can be obtained:

$$\frac{\partial\Delta_n}{\partial m}=\frac{\partial\Delta_n}{\partial I_i}\cdot\frac{\partial I_i}{\partial h_i}\cdot\frac{\partial h_i}{\partial m} \tag{17}$$

To summarise, to reduce the mid-span deflection by increasing the section depth of the beam, the most efficient position to add material is where the absolute value of $\partial\Delta/\partial m$ calculated by Eq. (17) is the largest among all sections along the span.

For example, for steel-reinforced rectangular sections, by combining Eqs. (13) and (15)-(17), it can be obtained:

$$\frac{\partial\Delta_n}{\partial m}=\frac{\partial\Delta_n}{\partial I_i}\cdot\frac{\partial I_i}{\partial h_i}\cdot\frac{\partial h_i}{\partial m}=-\frac{M_i}{2b\rho \mathrm{E}{I_e}^2}\left(\frac{L}{2}-l_i\right)\cdot\frac{\partial I_e}{\partial h_i} \tag{18}$$

The proposed method is quite general for statically determinate structures and is applicable to a wide range of scenarios, because:

- For other shapes of section (such as circular sections, pipe sections and I sections), Eq. (17) can be adjusted accordingly whist $\partial\Delta/\partial I_i$ will not change;
- For composite sections, the I_i of the transformed section can be used;
- For cracked sections in materials other than steel-reinforced concrete, as long as the theoretical expression of the effective moment of inertia I_e exists, the proposed method is still applicable.

3. Optimisation example

As shown in Fig. 4, a simply supported beam of 6 m long is under the linear load *w*. The design for ULS requires that the bending resistance of the beam should be sufficient under *w* = 32.8 kN/m, whilst the design for SLS requires that the maximum deflection of the beam should not exceed 1/250 of the span under *w* = 18.4 kN/m (the values of *w* is obtained from load combinations assuming the dead load G = 4 kN/m^2, the live load Q = 2 kN/m^2 and the beam spacing is 4 m). The concrete is C25/30 (f_{ck} = 25 MPa, $\mathrm{E_{cm}}$ = 31 GPa), and B500A steel bars are used for the longitudinal reinforcement (f_{yk} = 500 MPa, $\mathrm{E_s}$ = 210 GPa).

An initial shape is obtained from the design for ULS, as shown in Fig. 4a. The cross-sections in the mid-span and at the end of the span are shown in Fig. 4b and 4c. 3Φ20 B500A steel bars are used along the whole span, and the height of the beam varies from 400 mm to 200 mm. The section width is constant (200 mm), and the section height is designed in the way such that the bending resistance of each section is sufficient to resist the bending moment calculated from the ULS load combination (*w* = 32.8 kN/m). The minimum section height is fixed to 200 mm near the supports for practical reasons. The shear reinforcement is used to ensure the shear resistance, which is out of the scope of this study.

A first calculation of mid-span deflection shows the beam with the initial shape obtained from the design for ULS cannot satisfy the requirement for deflection (the deflection calculation is based on the method of ACI 318-19 [8] which takes into account the cracking of sections). Therefore, the optimisation for SLS should be conducted to reduce the maximum deflection under *w* = 18.4 kN/m.

The normalised influence lines obtained respectively by Eqs. (18) and by the method in [1] are shown in Fig. 4d. It can be seen that, according to the proposed method, the most sufficient position to add material is the mid-span section, whilst the method of [1] leads to rather different position far from the mid-span.

To reduce the maximum deflection, the material is to be added iteratively to the beam. First, the beam is divided into 100 segments (101 sections), and the influence line is obtained by Eq. (18). Subsequently, a unit height of 1 mm is added to the section where the influence line reaches its maximum. Then, the deflection with the new shape is calculated. If it still exceeds the maximum allowable value, the influence line is updated and a new iteration starts, until the deflection satisfies the requirement.

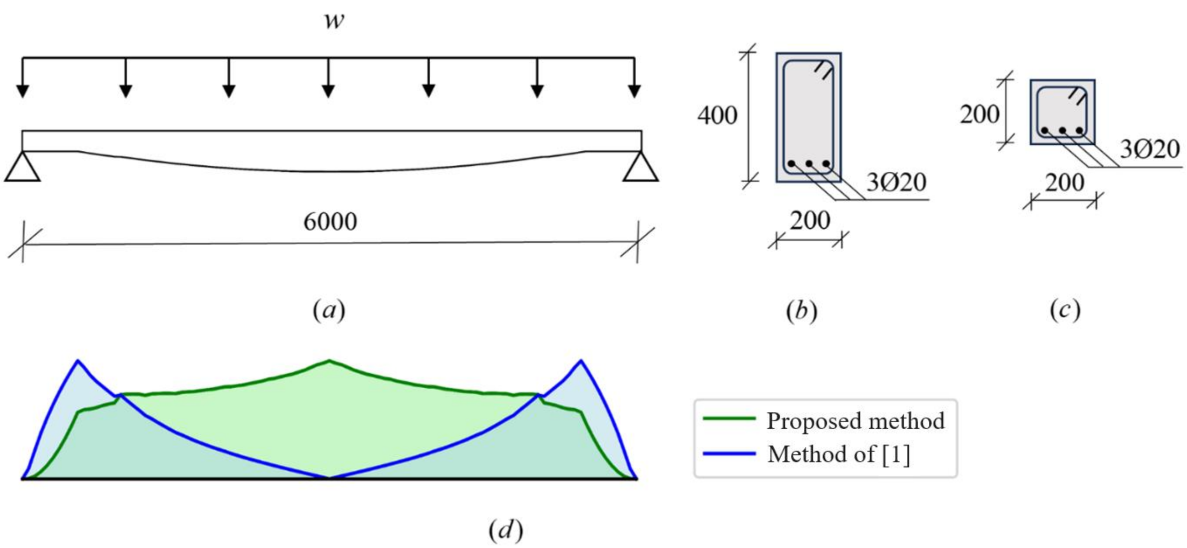
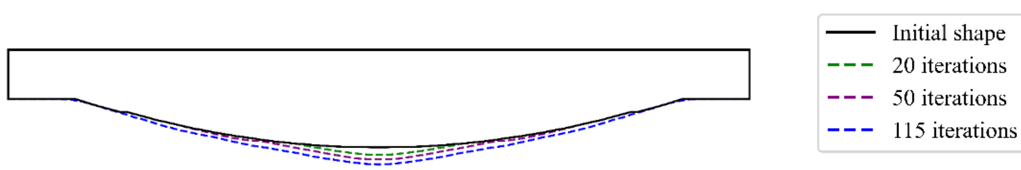


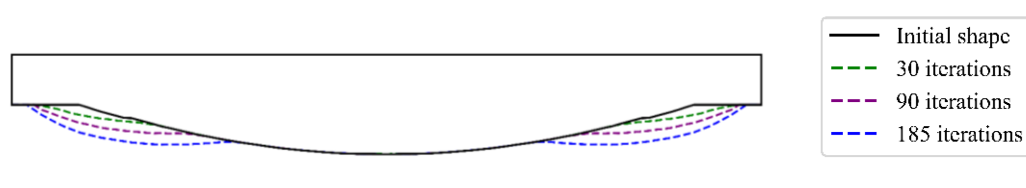
Fig. 4: An example of the optimisation of statically determinate beam under uniformly distributed loads (a: initial shape of the beam obtained from ULS design; b: mid-span section; c: section at the end of the span; d: comparison of the normalised influence line with the initial shape using two different methods)

The optimisation results are shown in Fig. 5. With the proposed method, the first 20 iterations mainly add material to the region near the mid-span, after which the addition of material comes gradually to the region further and further from the mid-span, as shown in Fig. 5a. After 115 iterations in total, the deflection no longer exceeds 1/250 of the span.

In contrast, with the method in [1], the material is added to the regions near the support, which requires more material than the proposed method to reach the same deflection, as shown in Fig. 5b. After 185 iterations in total, the deflection no longer exceeds 1/250 of the span. The proposed method adds 38% less material than the method in [1] and shows a higher efficiency.



(a) Proposed method



(b) Method of [1]

Fig. 5: Comparison of shape optimisation results (the beam height scaled up by 2 for visualisation)

4. Conclusion and discussion

The analytical method of shape optimisation of statically determinate beams for deflection control is proposed in this paper, and the rigorous mathematical derivation is given. The integration method for symmetric cases and the method of virtual power for unsymmetric cases lead to the same result, which is general for statically determinate structures, and can be adapted for those with different shapes of section, multiple types of material and cracked sections whose mechanical behaviour is not necessarily linear.

It can be argued whether the analytical method is still necessary, since the analysis model used in the process of shape optimisation can be numerical ones. For example, FEM (Finite Element Modelling) can be used to evaluate the optimal section to add material. However, due to the fact that the process of shape optimisation usually involves an enormous number of iterations, the analytical method can help save considerably computational costs and time compared to running a full numerical simulation for each iteration.

To go further, analytical methods for reducing the deflection of statically indeterminate structures can be investigated. A probable solution is to discretise the structure into segments and use the displacement method. The influence of the addition a unit mass on the stiffness matrix can be evaluated for sections at different positions. However, the assembly and the solving of the stiffness matrix may need involving numerical tools, reducing the advantage of using analytical method compared to full numerical methods. In fact, the more complex the structure is, the less straightforward obtaining the analytical solution would be, and the more convenient full numerical methods would become.

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