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Effect of Stochastic Randomness in Natural Frequency Increment of a Trapezoidal Laminated Composite Plate

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Abstract - This paper highlights the considerable stochastic effects while taking a deterministic approach to satisfy a dynamic objective function involving the free vibration behavior of a laminated composite trapezoidal plate. The study considers relevant material and geometric parameters of the plate under investigation as stochastic without changing the plan area of the plate which generally happens in many industrial and engineering applications. Investigation of the stochastic effects of material properties, ply thickness, and ply orientation on the objective of reaching an increased natural frequency is carried out by Monte Carlo simulation. Abaqus CAE is used to calculate the increment in the thickness required for the desired rise in the natural frequency of a symmetrically laminated plate. It is shown that variation in reaching the desired frequency increases with the increase of the coefficient of variation of the input variables. A positive linear trend between the coefficients of variation of the desired frequency and input variables is found, whose slope increases as randomness is considered for a greater number of input variables. Probability of failure in reaching the desired frequency is shown to be heavily dependent on the combined variation of material and geometry. In particular, the probability of failure in reaching a desired frequency shows an undulating trend for both types of variation considered. In general, it is also shown that randomness in geometry has a more pronounced effect compared to that in material only.

***Keywords*:** Non-rectangular trapezoidal plate, Dynamic properties, Increment in natural frequency, Material and geometric randomness, Stochastic effects

1. Introduction

Laminated composite materials are widely applied in many industries for their high strength-to-weight ratios and flexibility in design. Their ubiquity, especially in aerospace and aeronautical engineering, such as aircrafts, UAVs, helicopters, missiles, space stations including their use in fabricating civil engineering cladding units requires designers to have a thorough understanding of their mechanical behaviour. However, due to the uncertainty introduced in determining the strength and stiffness properties of laminates during the manufacturing of the individual composite layers and the laminated structure as a whole, a purely deterministic study can potentially be non-conservative and insufficient. Hence all such material and geometric properties shall be treated as random variables and their uncertainty should be quantified either experimentally or computationally.

Experimental data on the mechanical properties of unidirectional glass/polyester showed a coefficient of variation (CV) ranging between 10% to 20% for elastic and shear moduli as well as the material strengths, with variation as high as 24.90% [1]. In unidirectional carbon fibre-reinforced polymers (CFRP), experimental data showed variation as high as 13.1% for tensile strengths [2], with less variation in other material properties. Recent research [3] has shown that, while the elastic properties of carbon fibre/epoxy composites possess a CV of around 5%, the CV of the mechanical strength still ranges from 10% to 20% [4,5]. Uncertainties in material properties lead to uncertainties in dynamic behaviour [6].

Natural frequency enhancement to avoid resonance due to machine installation is often required within a restricted plan area. The geometric and material properties always have elements of uncertainty as a natural occurrence due to the inevitable fabrication inaccuracies in layup and curing. In such cases, even a slight shift in the characteristics of any of the plate properties can have a pronounced effect on the response of the structure. This study aims to predict the stochastic effects that one faces while dealing with such problems arising due to randomness in material and geometric properties.

2. Brief Overview

The determination of natural frequency is done with the help of Finite Element based software Abaqus CAE using Mindlin-Reissner Plate theory.

2.1. Objective function

A trapezoidal clamped [0º/90º/0º] plate of the following dimensions and properties as presented in Table 1 and Table 2, is modelled in Abaqus CAE. A deterministic study is done to note the additional thickness that is needed to have ten percent increase of natural frequency.

Table 1: Model used in the study.

Longer side	4000 mm
Shorter side	3000 mm
Width	3000 mm
Thickness	90 mm
Thickness of individual lamina	30 mm each
Natural frequency	0.542455 Hz
Desired natural frequency (i.e. ten percent increase)	0.596704 Hz
Thickness required per lamina to attain the desired natural frequency	33.143 mm

Table 2: Mean material properties and parameters.

Material	E_{11}(GPa)	E_{22}(GPa)	Poisson's ratio (ν)	G_{12}(GPa)	G_{13}(GPa)	G_{23}(GPa)	θ(°)	t(mm)
Glass/epoxy	38.6	8.27	0.26	4.14	4.14	4.14	0º / 90º	33.143

2.2. Frequency ratio

By the objective function described above, reaching the desired frequency is a Boolean (i.e., 'true' or 'false') value, but they provide very little information about how much the thickness can be decreased or increased if the laminate yields a frequency greater or lesser than the desired frequency respectively. Thus, a frequency ratio (fr) is introduced and is accordingly used to compare all objective function failure scenarios in this study and the ratio is simply a metric defined as:

$$\mathrm{fr} = \frac{\nu_{desired}}{\nu_{model}} \tag{1}$$

where ν_{model} is the frequency of the model and $\nu_{desired}$ is the ten percent increased frequency.

3. Brief overview of technique and statistical metrics

3.1. Monte Carlo simulation

In this study, Monte Carlo simulation (MCS), a simple computational approach to stochastic physical problems involving many degrees of freedom, is used to generate the data. To obtain numerical results, MCS involves repeated sampling of random variables to simulate an arbitrarily large number of experiments [7]. For each trial in this study, the material properties, material strengths, ply orientations and ply thicknesses form a vector **X** of basic random variables and is given by:

$$\mathbf{X} = [\,(E_{11}\ \ E_{12}\ \ G_{12}\ \ \theta\ \ t)_1 \quad (E_{11}\ \ E_{12}\ \ G_{12}\ \ \theta\ \ t)_2 \quad (E_{11}\ \ E_{12}\ \ G_{12}\ \ \theta\ \ t)_3\,] \tag{2}$$

All material and geometric properties are allowed to vary at the ply level, but for simplicity G_{13} and G_{23} are assumed to be equal for all plies in each laminate. In this study, with 15 stochastic degrees of freedom, the vector **X** in Eq. (2) highlights highlights the increasing complexity and uncertainty of the laminate frequency ratio as a random variable.

3.2. Probability of failure and correction factor

The probability of failure is defined in this study as the proportion of laminates in a simulation of N trials whose resulting frequency is less than the predicted deterministic value $\nu_{desired}$ and can be calculated as:

$$P_f = \frac{1}{N}\sum_{i=1}^{N} I(\nu_{model_i} < \nu_{desired}) \tag{3}$$

where $I(\nu_{model_i} < \nu_{desired})$ is the indicator of the objective function to reach the desired frequency, defined as:

$$I(\nu_{model_i} < \nu_{desired}) = \begin{cases} 1\ if\ \nu_{model_i} \leq \nu_{desired} \\ 0\ if\ \nu_{model_i} > \nu_{desired} \end{cases} \tag{4}$$

and the frequency of the i^{th} model ν_{model_i} is the manifestation of the random variable ν_{model} defined as a function of the random variable **X** in trial i, i.e. $\nu_{model_i} = \nu_{model}(\boldsymbol{X}_i)$.

Defining failure probability this way would neither adequately reflect to what extent the objective is achieved nor would indicate the extent to what the objective remains unattained. To address this, a correction factor (CF) is defined as:

$$\mathrm{CF} = \frac{\nu_{desired}}{\nu_{model_{failed}}} \tag{5}$$

where $\nu_{model_{failed}}$ is the frequency of the model of the failed laminate (i.e. $\nu_{model_{failed}} = \nu_{model_i}$ when $\nu_{model_i} \leq \nu_{desired}$). Hence, CF is defined only when the laminate fails. CF can be interpreted as the factor by which the frequency of a laminate must be multiplied to reach the predicted desired deterministic value, thus acting as a tool to compare trials.

3.3. The empirical distribution function and sample quantiles

The cumulative distribution function (CDF) of a random variable, Y, is used to describe the distribution of that random variable and is defined as:

$$F_Y(y) = \mathrm{P}(Y \leq y), \qquad y \in R \tag{6}$$

More importantly, the CDF effectively contains all information about the random variable and completely determines the shape of its distribution. The CDF can be approximated by:

$$\hat{F}_Y(y) = \frac{1}{N}\sum_{i=1}^{N} I(Y_i < y) \tag{7}$$

which may be defined here as the empirical distribution function (EDF). The EDF of the frequency can therefore be defined as:

$$\hat{F}_v(x) = \frac{1}{N}\sum_{i=1}^{N} I(v_{model_i} < x), \qquad min\{v_{model_i}\} \leq x < max\{v_{model_i}\} \tag{8}$$

where min $\{v_{model_i}\}$ and max $\{v_{model_i}\}$ are the lowest and highest realizations of the frequencies of all trials in each simulation.

4. Simulation set up

4.1. Procedure

Abaqus CAE is used to compute the frequencies of various combinations of laminates. Frequency data are generated using MCS for two different simulations which consider combinations of material and laminate geometric parameters to explore the interaction effects of randomness in each type of simulation. Five different coefficients of variation are taken up for each simulation. Each simulation is repeated for 100 trials, resulting in a total of 500 simulations for each combination of material properties and another 500 simulations for each combination of material and geometric properties. Each time, the model is reset to eliminate the interaction effects, if any, between simulations. Table 3 provides a summary.

Table 3: List of simulations performed.

Simulation	Random parameters
1	Material properties (E_{11}, E_{22}, G_{12})
2	Material and geometric properties ($E_{11}, E_{22}, G_{12}, \theta, t$)

Materials examined and their corresponding mean values are summarized in Table 2. These materials were used in [8] as well. The standard deviations of all material properties are defined functions of their mean values and CV. An exception is made for the ply orientation, whose standard deviation is assumed to have a maximum of 1.8° at the upper bound of simulated CV values and varies linearly with CV (e.g., the deviation is 0.45° for CV = 0.05) as in [6].

Experimental data from Maekawa [2] regarding material properties of unidirectional carbon fibre-reinforced laminates shows that the distribution of basic material parameters can be closely approximated by a Gaussian or normal distribution. Using the Kolmogorov-Smirnov test, Lekou and Philippidis [1] showed that the assumption of normally distributed mechanical properties for unidirectional glass/polyester cannot be rejected at 5% significance level. Thus, the assumptions of normally distributed material properties, strengths, ply thicknesses and ply orientations for the present analysis are valid.

4.2. Model validation

The element and the mesh that is used in the study is first benchmarked with the non-dimensional frequency (ω_{nd}) of a clamped graphite-epoxy composite $[0^o/90^o]_4$ square plate model (235 mm x 235 mm and 2.8 mm thick) presented by Sinha et al. [9]. With a plan dimension of 'b', the fundamental frequency (ω) is made non-dimensional as:

$$\omega_{nd} = \omega b^2 (\rho / E_{22} h^2)^{1/2} \tag{9}$$

ω_{nd} values of two different problems with two and three unidirectional stiffeners (each stiffener 11.2 mm wide and 16.8 mm deep) respectively are calculated by the present approach and furnished in Table 4. The results show a very close agreement for a mesh of 50x50. With this order of finite element mesh, the simulations in this study are performed.

Table 4: Non-dimensional fundamental frequencies (ω_{nd}) of square plate used for benchmarking.

Model	Sinha et al. (ref. [9])	Present study	Percent deviation
Two stiffeners along x direction	35	35.019	0.054
Three stiffeners along x direction	37	37.069	0.186

5. Results and discussion

5.1. Frequency coefficient of variation

Fig. 1 shows monotonic increase in frequency coefficient of variation (CV(ν)) with input coefficient of variation (CV(X)) for the symmetric trapezoidal composite laminates considered in this study. The linear trend-line and polynomial fit-curve are also shown alongside. It is noted that a three-degree polynomial fits, which signifies that the coefficient of variation is not linear. It is apparent that the linear increase in natural frequency and displacement coefficients of variation as reported in ref. [10] is not mirrored in this study which shows a clear non-linear dependence of the frequency coefficient of variation with variation of the input variables.

Across all laminates and materials considered in the present study, the average slope is 0.2331 when considering all material properties as random and a slope of 0.6379 considering all material and geometric properties as random. This indicates that across all simulations, each unit change in CV(X) results in disproportionate change in the CV(ν). The greatest effect on CV(ν) is due to variations of geometric properties particularly thickness as reported in references [6] and [11]. The importance of thickness was also reported by Gohari et al. [12] who found that slight changes in shell lay-up thickness caused considerable fluctuations in failure strength. In the present study, similar observation of non-linear dependence is observed in frequency related study as well.

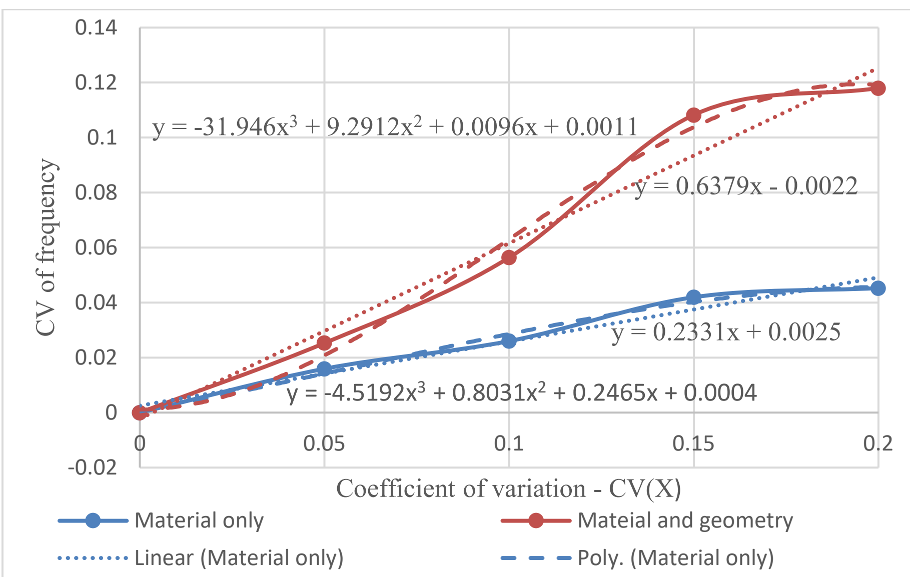

Fig. 1: Coefficients of variation: Frequency vs Input variables.

5.2. Probability of failure

Fig. 2 shows the probability of failure for the two considered combinations of randomness. It is concluded that this metric is generally non-linear and of an overlapping nature over its range of coefficients of variation in input variables. Whether a combination of material and geometric randomness has more impact than only material randomness in the failure probability cannot be concluded. However, the study suggests the range in which the variation will occur. At higher values of variation coefficient, the failure probability takes a leap for those cases where both material and geometric properties are varying randomly, which suggests that extra attention should be paid for geometric variations.

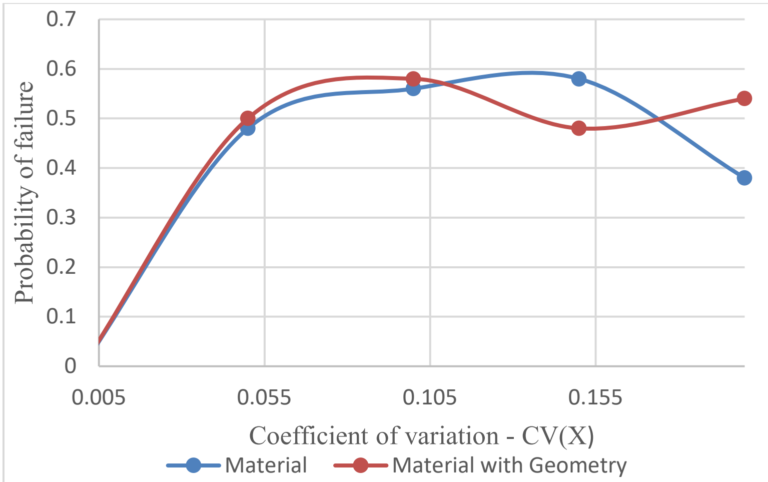

Fig. 2: Probability of failure: Simulation comparison.

5.3. Correction factor

Fig. 3 shows a strong, positive correlation between the mean correction factor and CV(X). The mean is preferred over the median when dealing with data containing outliers because it is more sensitive to extreme values, making it a more appropriate measure of central tendency. It shows, if not exact, an approximate linear behaviour for both types of simulations. However, with higher values of variable X coefficient of variation, it is noticed that the correction factor for the simulation where both material and geometry are varied rises progressively.

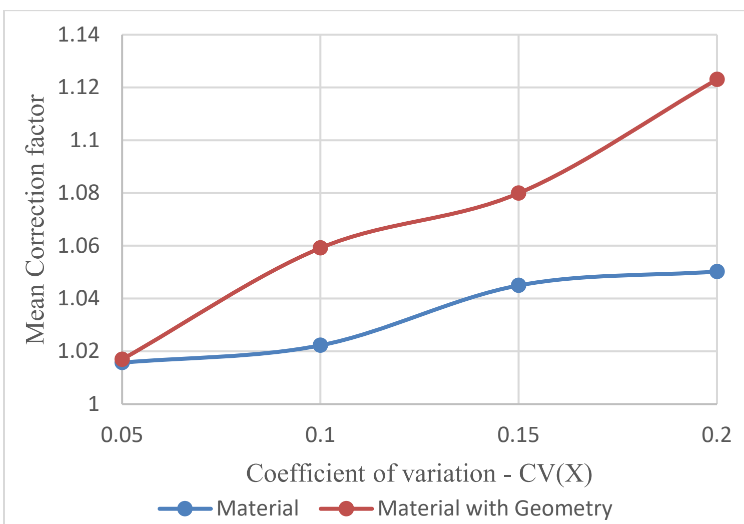

Fig. 3: Correction factor: Simulation comparison.

Figs. 4 and 5 show the EDF of the correction factor for both types of simulations respectively. The EDFs clearly illustrate a notable rise in sample quantiles of safety factors as the coefficient of variation of variable X increases. For second simulation of Glass/Epoxy (Fig. 5), 90% of simulated laminates have a correction factor less than or equal to 1.04 when

CV(X) is 0.05 but is 1.132 when CV(X) is 0.15. In other words, with CV = 0.05, 90% of laminates must be manufactured 4% thicker to ensure attainment of the expected deterministic frequency, compared to 13.2% when considering CV = 0.15. These results agree with the findings of Khasaba et al. [13]. It is also noted from the two figures that the addition of randomness in geometry variation incurs more penalty than the variation in material property alone.

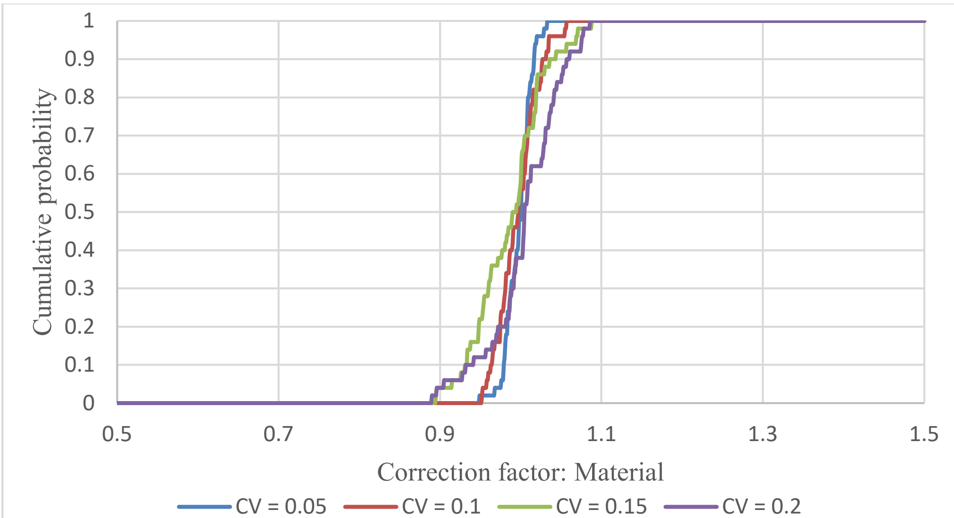

Fig. 4: Correction factor EDF: Material as random variable.

Fig. 5 illustrates how the addition of geometric randomness incurs more correction factor for the same cumulative probability.

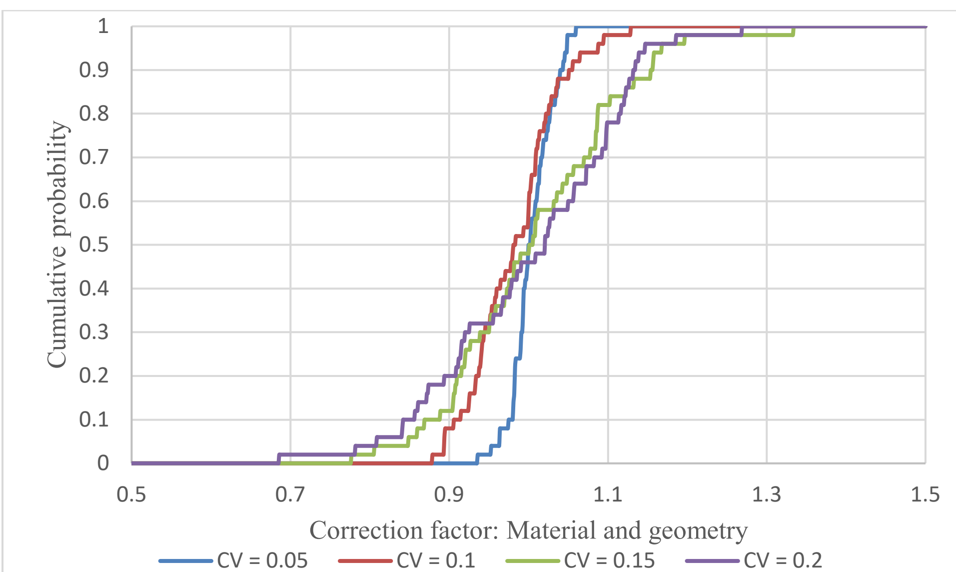

Fig. 5: Correction factor EDF: Material and geometry as random variable.

4. Conclusion

Natural frequencies predicted by deterministic analysis are often inaccurate when considering randomness in material properties and laminate parameters. This paper:

- Underscores the significance of accounting randomness in achieving a targeted natural frequency by augmenting thickness in a symmetric ply when the plan area is kept unaltered which is generally encountered in practical situations.
- A non-linear relationship exists between coefficient of variation of all input variables and frequency, but it is closer to linear for mean correction factor. Probability of failure is generally nonlinear.
- The disproportionate sensitivity of the coefficient of variation of natural frequency (CV(ν)) to the coefficient of variation of all material and geometric parameters (CV(X)), particularly focusing on ply thickness, accentuates the importance of incorporating more manufacturing accuracy.
- Overall, the penalty incurred to ensure that a laminate aligns with the anticipated deterministic frequency escalates more with the coefficient of variation (CV) of variable X when geometric parameters also vary randomly.

These findings highlight the importance of considering randomness in every available factor of design of composite laminates like stacking sequence, material selection and particularly thickness control in the manufacturing process.

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