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Simulation of the Backward-facing Step Flow Using the Meshless Local Petrov-Galerkin Method

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Abstract- The paper deals with use of the meshless method for incompressible fluid flow analysis. There are many formulations of the meshless methods. The article presents the Meshless Local Petrov-Galerkin method (MLPG) – local weak formulation of the Navier-Stokes equations. The shape function construction is the crucial part of the meshless numerical analysis in the construction of shape functions. The article presents the radial point interpolation method (RPIM) for the shape functions construction.

Keywords**:** meshless analysis, meshless Petrov-Galerkin method, Navier-Stokes equations, numerical simulation

1. Introduction

Incompressible Navier-Stokes flow in two dimensions is one of the several major problems in fluid mechanics that have been extensively studied both theoretically and numerically. In general, the formulation of incompressible Navier-Stokes equations using primitive variables is often used, but it has limitation in approximating the velocity and pressure. The meshless Local Petrov-Galerkin method (MLPG) is truly meshless method which requires no elements or global background mesh, for either interpolation or integration purposes. The first article applying MLPG method to compute convection-diffusion and incompressible flow was by Lin and Atluri (2001). In their work, two kinds of upwind schemes were constructed to overcome oscillations produced by convection term. They applied the upwind schemes to solve incompressible flow problem based on primitive variables formulation and added the perturbation term to the continuity equation to satisfy Babuška-Brezzi condition. But there still persists the problem of perturbation parameter determination for high Reynolds number problems. The present paper focuses on the MLPG primitive variable method using fractional step method to achieve velocity-pressure decoupling to solve incompressible viscous flow (Sataprahma and Luadsonga, 2013).

2. Governing Equations and Fractional-Step Algorithm

The governing equations for unsteady incompressible viscous fluid flow are Navier-Stokes equations with the continuity equation in the convection term (Sataprahma and Luadsonga, 2013). This equation can be written as

$$\nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_j}\left(u_j u_i\right) - \frac{\partial u_i}{\partial t} = f_i \qquad (1)$$

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad (2)$$

Where u_i is the velocity in direction i, p is the pressure, f_i is the body force component, ρ is a density of a liquid and ν is the kinematic viscosity. Eq.(1) is the momentum equation and Eq.(2) is the continuity equation. A fractional-step algorithm is used to solve this problem (Kovarik et al., 2014). The time derivative of the velocity vector in a momentum Eq.(1) can be replaced with a difference approximation and following relation is obtained

$$u_i^{n+1} = u_i^n + \Delta t \left[\nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} - f_i - \frac{\partial}{\partial x_j}\left(u_j u_i\right) + \frac{\Delta t}{2} u_k \frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_j}\left(u_j u_i\right) + f_i\right)\right]^n - \frac{\Delta t}{\rho}\frac{\partial p^{n+1}}{\partial x_i} \tag{3}$$

Where upper indexes n and n+1 indicate the time step. Eq.(3) is explicit formula for convection and viscous terms and the implicit one for a pressure term. To simplify Eq.(3) we used the fractional step approximation (Kovarik et al., 2014). According this approximation, the intermediate velocity $\tilde{u}_i$ components are computed using simplified momentum equation

$$\tilde{u}_i = u_i^n + \Delta t \left[\nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} - f_i - \frac{\partial}{\partial x_j}\left(u_j u_i\right) + \frac{\Delta t}{2} u_k \frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_j}\left(u_j u_i\right) + f_i\right)\right]^n \tag{4}$$

When we compare Eq.(3) and Eq.(4) we get

$$\mathrm{u}_\mathrm{i}^{\mathrm{n}+1} = \tilde{\mathrm{u}}_\mathrm{i} - \frac{\Delta \mathrm{t}}{\rho}\frac{\partial \mathrm{p}^{\mathrm{n}+1}}{\partial \mathrm{x}_\mathrm{i}} \tag{5}$$

The intermediate velocities $\tilde{u}_i$ does not satisfy the continuity equation (Eq.(2)). The velocities u_i^{n+1} must satisfy the continuity equation which implies

$$\frac{\partial^2 p^{n+1}}{\partial x_i \partial x_i} = \frac{\rho}{\Delta t}\frac{\partial \tilde{u}_i}{\partial x_i} \tag{6}$$

Eq. (6) is the Poisson's equation with non-zero source term (Kovarik et al., 2012). The pressure Eq.(6) is solved using MLPG over problem domain with boundary conditions $p^n|\Gamma_u = \bar{p}^n$ and $\partial p^n/\partial n = \bar{q}^n$.

3. The MLPG Method and the Local Weak Formulation

The meshless Local Petrov-Galerkin method (MLPG) is truly meshless method which requires no elements or global background mesh, for either interpolation or integration purposes. In MLPG the problem domain is represented by a set of arbitrarily distributed nodes (Kovarik, 2011).

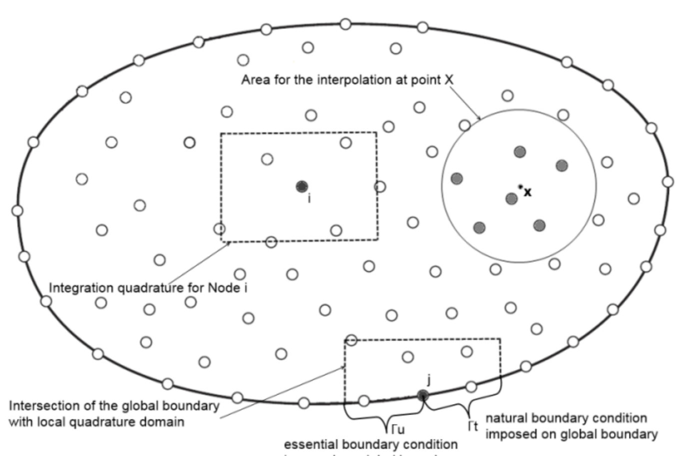

Fig. 1. Schematic of local quadrature domain, essential and natural interested boundary.

The weighted residual method is used to create the discrete system equation by integrating the governing equation over local quadrature domains (see. Fig.1). The quadrature domain can be arbitrary in theory, but very simple regularly shaped domain, such as rectangles for 2D problems are often used for ease of implementation (Izvoltova and Villim, 2012).
A generalized local weak form of the pressure Poisson Eq.(6) defined over local sub-domain Ω_s can be written as

$$\int_{\Omega_s} \left[\frac{\partial^2 p^{n+1}}{\partial \mathrm{x_i}\, \partial \mathrm{x_i}} - \frac{\rho}{\Delta \mathrm{t}} \frac{\partial \tilde{\mathrm{u}}_\mathrm{i}}{\partial \mathrm{x_i}}\right] \mathrm{w} d\Omega = 0 \tag{7}$$

Where p is pressure, w is the test function defined as

$$w(r_i) = \begin{cases} 1 - 6r_i^2 + 8r_i^3 - 3r_i^4 & r_i \le 1 \\ 0 & r_i \succ 1 \end{cases} \quad ; \qquad r_i = \frac{\|x - x_i\|}{d_s} \tag{8}$$

Where d_s is the size of the local quadrature domain, so it is evident that weighting function value is zero on its boundary. The choice if this test function is motivated by its ability to vanish on the boundary of local quadrature domain. Using the divergence theorem the Eq.(7) has changed to

$$\int_{\Omega_s} \frac{\partial p^{n+1}}{\partial x_i} \frac{\partial w}{\partial x_i} d\Omega - \int_{\Gamma_{su}} \frac{\partial p^{n+1}}{\partial x_i} w n_i d\Gamma = \int_{\Gamma_{sq}} \frac{\partial p^{n+1}}{\partial x_i} w n_i \, d\Gamma - \frac{1}{\Delta \mathrm{t}} \int_{\Gamma_{su \cup sq}} \tilde{u}_i^n \mathrm{n_i w} d\Gamma + \frac{1}{\Delta \mathrm{t}} \int_{\Omega_s} \tilde{\mathrm{u}}_\mathrm{i}^\mathrm{n} \frac{\partial \mathrm{w}}{\partial \mathrm{x}} d\Omega \tag{9}$$

Because unknown nodal values of the pressure p, are constants which can be moved out of the integral the equation, Eq.(9) can be changed to discrete system of linear equations, where global “stiffness” matrix is defined as

$$K_{mn} = \int_{\Omega_s^m} \frac{\partial \phi_n}{\partial x_i} \frac{\partial w_m}{\partial x_i} d\Omega - \int_{\Gamma_{\mathrm{su}}^\mathrm{m}} \frac{\partial \phi_\mathrm{n}}{\partial \mathrm{x_i}} \mathrm{w_m n_i} d\Gamma \tag{10}$$

and the right-hand side “load” vector is

$$f_m = \int_{\Gamma_{sq}^m} \bar{q} w_m \, d\Gamma - \frac{1}{\Delta t} \int_{\Gamma_{su \cup sq}^m} \tilde{u}_i^n n_i w_m d\Gamma + \frac{1}{\Delta t} \int_{\Omega_s^m} \tilde{u}_i^n \frac{\partial w_m}{\partial x_i} d\Omega \tag{11}$$

The term φ in Eq.(10) represents the trial function, in this case the Multi-Quadrics Radial Basis function (MQ-RBF), details can be found in Kovarik (2011). The weak form of the equations Eq.(4) and Eq.(5) can be written as follows, assuming for simplicity $f_i = 0$

$$\int_{\Omega_s} w \tilde{u}_i \, d\Omega = \int_{\Omega_s} w u_i^n \, d\Omega + \Delta t \int_{\Omega_s} w \left[\nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}(u_j u_i) + \frac{\Delta t}{2} u_k \frac{\partial}{\partial x_k} \left(\frac{\partial}{\partial x_j}(u_j u_i) \right) \right]^n d\Omega \tag{12}$$

$$\int_{\Omega_s} w u_i^{n+1} \, d\Omega = \int_{\Omega_s} w \tilde{u}_i \, d\Omega - \frac{\Delta t}{\rho} \int_{\Omega_s} \frac{\partial p^{n+1}}{\partial x_i} d\Omega \tag{13}$$

After application of divergence theorem and eliminating boundary integrals over which the test function vanishes, the following form is obtained

The mass matrix in Eq.(13) a Eq.(14) can be used in either a consistent or lumped form. The lumped form is used here, because it eliminates a matrix inversion procedure. The fractional time step algorithm described above is now used to solve the Navier-Stokes equations at every time step (Kovarik et al., 2014):

- Step 1: Computation of the intermediate velocity $\tilde{u}$ from the velocities at the previous time step using Eq. (14),
- Step 2: Solution of pressure Poisson equation Eq.(9),
- Step 3: Computation of velocities at the current time step from Eq.(14).

$$\int_{\Omega_s} w\tilde{u}_i \, d\Omega = \int_{\Omega_s} w u_i^n \, d\Omega + \Delta t \begin{bmatrix} \nu \int_{\Omega_s} \frac{\partial^2 u_i}{\partial x_j \partial x_j} d\Omega - \int_{\Omega_s} w \frac{\partial}{\partial x_j}\left(u_j u_i\right) d\Omega + \\ + \frac{\Delta t}{2} \int_{\Omega_s} w \frac{\partial u_k}{\partial x_k}\left(\frac{\partial}{\partial x_j}\left(u_j u_i\right)\right) d\Omega + \int_{\Omega_s} \frac{\partial w}{\partial x_k} u_k \left(\frac{\partial}{\partial x_j}\left(u_j u_i\right)\right) d\Omega \end{bmatrix}^n \tag{14}$$

4. Numerical Example - backward-facing step flow

The flow over a backward-facing step is a widely tested configuration of fluid flow in a channel. The geometry and boundary conditions used here are chosen to be similar to those of Najafi et al. (2012) or Erturk (2008) to facilitate comparison (see Fig. 2). The inlet velocity is assumed to be horizontal with a parabolic distribution, and the value of the horizontal velocity component is computed as (Kovarik et al., 2014)

$$\mathrm{u} = 24(\mathrm{y} - 0.5)(1 - \mathrm{y}) \tag{15}$$

The maximum inflow velocity is u_{max} = 1.5, and the average inflow velocity is u_a=1. The Reynolds number can then be defined as

$$\mathrm{Re} = \frac{\mathrm{u_a H}}{\nu} \tag{16}$$

Where H is the height of the channel (see Fig. 2).

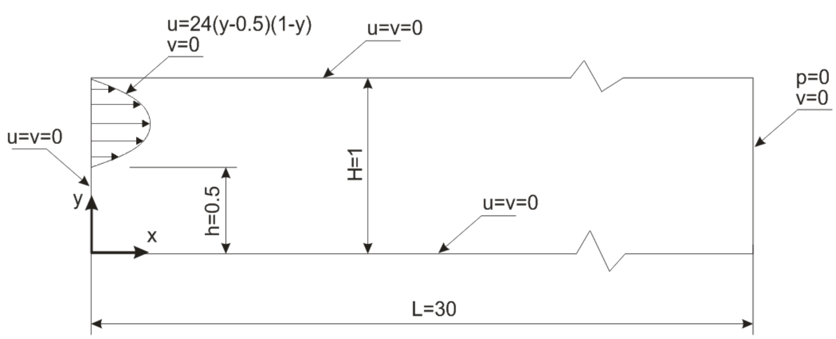

Fig. 2 Backward-facing step, geometry and boundary conditions.

The whole domain is covered by two uniform nodal models. The first consists of 121 points in the x direction and 21 in the y direction for $0 \le x \le 12$; the second consists of 180×11 points for $12 \le x \le 30$. The initial values of all quantities are set to zero.

The steady solution is again reached when the tolerance between two consecutive time steps (Eq.(13)) is lower than a prescribed tolerance value ε. Fig. 3 shows streamlines and pressure contours for Re=800, and Fig. 4 compares the horizontal velocity components in two vertical profiles (x = 3 and x = 7), with values presented in Erturk (2008).

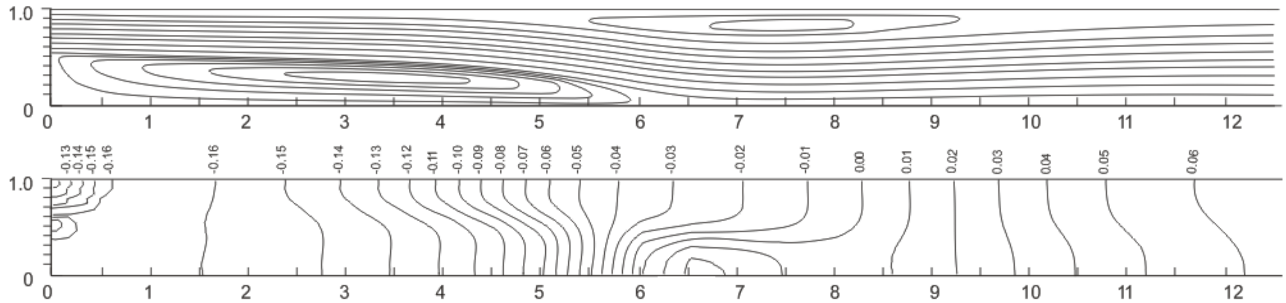

Fig. 3 Streamlines and pressure contours for Re = 800.

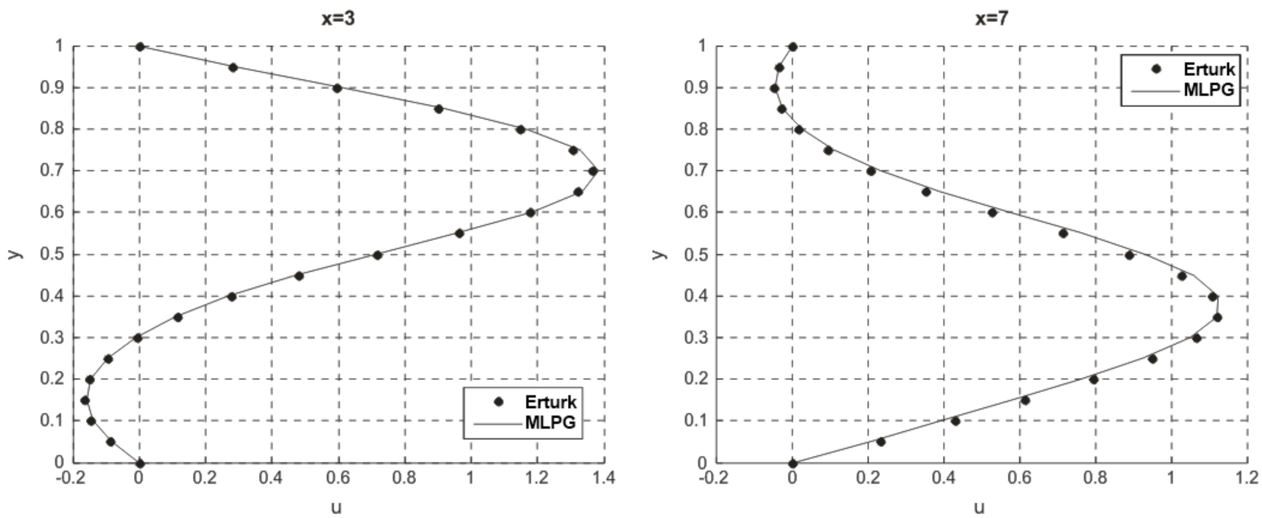

Fig. 4 Comparison of the horizontal velocities in profiles x = 3 and x = 7 for Re = 800.

The figure shows that at low Reynolds numbers our computed results agree well with both the other results. As the Reynolds number increases, the agreement with the experimental results becomes only moderate, because the flow in the experiments becomes three-dimensional, preventing direct comparison between the experimental and the 2D numerical results (Armaly et al., 1983).

5. Conclusion

In this article, a numerical algorithm using the Meshless Local Petrov-Galerkin (MLPG) method for the incompressible Navier-Stokes equations is demonstrated. To deal with convection term, the fractional step method was adopted and the set of recurrent equations was derived for time stepping procedure. The ability of the MLPG code to solve fluid dynamics problems was presented by solving backward-facing step flow problem with reasonable accuracy when compared to exact solution.

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