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Towards the Inviscid Limit: A New Perspective on TKE in Forced Burgers Turbulence

Kiarash Jalali
École de technologie supérieure
1100 Notre-Dame St W, Montréal, Canada
kiarash.jalali.1@ens.etsmtl.ca
Chestnut Conference Centre - University of Toronto, Toronto.ca

***Abstract* -** In the realm of fluid dynamics, understanding the details of turbulent kinetic energy (TKE) behavior under varying conditions remains crucial. This study revisits the classic problem of forced Burgers turbulence, specifically examining the influence of Reynolds number on TKE, in light of historical assertions that TKE is independent of Reynolds number variations (if they are large enough). Contrary to the long-standing claims made by D.T.Jeng and W.C.Meecham in 1972, our investigation discusses a nuanced relationship between Reynolds number and TKE. By employing sinusoidal forcing with a zero initial velocity field, we carefully analyse the transition of TKE as the system approaches the limit of vanishing viscosity. Our results unambiguously indicate that, with an increase in Reynolds number, TKE exhibits a convergence towards the inviscid solution, suggesting a dependence on viscosity that has not been fully acknowledged in previous studies. This perspective enriches our understanding of turbulence dynamics by highlighting the critical role of viscosity, as it diminishes. Our research serves as a cornerstone for re-evaluating theoretical approaches to turbulence, particularly in modelling and simulation frameworks that have probably overlooked the subtleties of viscosity's influence on TKE. By drawing attention to these complexities, we pave the way for more accurate predictions and a deeper comprehension of turbulent flows across a wide spectrum of applications.

***Keywords*:** Evolution of TKE, Forced Burgers turbulence, Periodic driving force, Direct Numerical Simulations (DNS), Fourier-Galerkin numerical method

1. Introduction

Burgers turbulence serves as a simplified model to study the dynamics of Navier-stokes turbulence, offering insights into energy transfer, shock wave formation, and the impact of viscosity on turbulent behaviour [1]. Historically, the study of steady state forced Burgers turbulence, especially in the context of TKE, has been pivotal in understanding these complex phenomena [2].

One of the longstanding assertions in the field, based on the seminal work by D.T. Jeng et al. is the supposed independence of TKE from the viscosity in the regime of high Reynolds numbers [3]. This assumption has underpinned numerous theoretical and computational models, shaping our understanding of turbulence dynamics. However, the relationship between viscosity (inversely related to the Reynolds number) and TKE, particularly in the limit of vanishing viscosity, remained inadequately explored and potentially misunderstood.

In light of these considerations, our investigation revisits the forced Burgers turbulence model with a critical eye towards the behaviour of TKE as viscosity approaches zero. Contrary to established claim, preliminary findings suggest that TKE may not remain independent of the Reynolds number. Instead, there appears to be a convergence towards the inviscid solution, indicating a more complex and dynamic relationship between Reynolds number and TKE than previously acknowledged.

Furthermore, a distinguishing aspect of our investigation is the enhanced resolution of our simulations, surpassing those of previous studies in capturing the intricate dynamics and the kinetic energy of the forced Burgers turbulence. This advancement not only facilitates a more precise analysis of TKE behaviour as viscosity diminishes but also brings forth a closer correspondence between the observed velocity field curves and its theoretical analytical counterpart. Such improvements in simulation fidelity are critical, as they allow for a more accurate and detailed exploration of turbulence phenomena, offering insights that were previously obscured by the limitations of computational resources. Through these methodological enhancements, our work not only challenges existing theoretical frameworks but also sets new benchmarks for the simulation and understanding of turbulent flows, paving the way for future research to build upon a more accurate and nuanced foundation of turbulence behaviour.

2. The Description of Numerical Simulation

We aim to determine the velocity $u(x,t)$ based on a specified initial velocity field and a sinusoidal forcing term. To achieve this, we initially introduce the mathematical definitions of the problem and the methodology employed in our research.

2.1. Mathematical definition

The one-dimensional Forced Burgers turbulence is modelled as follows

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} + \mathbf{u}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \boldsymbol{\nu}\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \mathbf{f} \tag{1}$$

In which $u = u(x,t)$ is the velocity field, the driving force f is a smooth function acts as a mechanism through which energy can be injected into the field. The outcome is a mechanism that introduces kinetic energy to the largest scales and transfers it, through the energy cascade due to nonlinear interactions, to the smallest scales where dissipation takes place.

We confine the forcing term to a sinusoidal function and focus primarily on analysing the steady state, which represents a state of equilibrium. Consequently, we suppose

$$\mathbf{f} = -\mathbf{A}\sin(\mathbf{kx} - \boldsymbol{\omega}\mathbf{t}), \quad \mathbf{A} > \mathbf{0} \tag{2}$$

For the purpose of simplifying the analytical solution, we can consider the inviscid $\nu = 0$ forced Burger's equation

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} + \mathbf{u}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \mathbf{f} \tag{3}$$

By substituting Eq. 2 in Eq. 3 and solving the equation, we obtain the analytical solution [3],[4].

$$\mathbf{u} = \mathbf{c} \pm \mathbf{c}(2\mathbf{a})^{1/2}\cos\frac{\boldsymbol{\xi}}{2} \tag{4}$$

This is confirmed by the force values, with $c = \omega/k$ representing the speed of the moving force and $a = \frac{2A}{kc^2}$, where k is the wavenumber. The cosine wave is defined across the periodic range of $-\xi_0 \leq \xi \leq 2\pi - \xi_0$. Within this range, a discontinuity at the shock position ξ_0 causes a sign change, indicating the shock position.

$$\boldsymbol{\xi}_0 = 2\sin^{-1}(\boldsymbol{\pi}^2/8\mathbf{a}) \tag{5}$$

2.2. Methodology

In line with the simulations by Jeng & Meecham [3], the calculations were carried out in an equilibrium state introducing the forcing term to the smallest wavenumber, largest scale at $k = 2\pi$ with $A = 10$ and speed of moving force equal to unity.

By adopting appropriate length scale L and velocity scale V which are both equal to unity and applying these dimensions to transform the model equation into a nondimensional form [2], we obtain

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} + \mathbf{u}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{\mathbf{1}}{\mathbf{Re}}\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \mathbf{f} \tag{6}$$

Given that $Re = VL/\nu$. The research, conducted across a range of Reynolds numbers $Re = 100, 250\ and\ 500$ aligns with the studies by Jeng and Meecham [3]. Besides we conducted this research work in higher Reynold number of $Re = 5000$ to evaluate a scenario closer to the inviscid limit.

The forced Burgers equation is numerically simulated in MATLAB software using a Fourier-Galerkin solver with a 3rd-order compact Runge-Kutta time integration scheme. The domain length is set over the range $0 \leq x \leq 1$ and the spectral resolution utilize up to $N = 24576$ Fourier modes, accounting for the de-aliased modes as well, and the temporal mesh size chosen to be up to $\delta t = 1e - 5$. To address the aliasing error, we implemented the two-third rule de-aliasing technique [5].

The forcing term is integrated into the velocity field and introduced to the model at the end of each time step. To maintain the stability, we evaluated the mesh size based on the local CFL condition [6].

This research was carried out in a state of equilibrium, where the energy injected by the forcing term is balanced by the energy dissipated due to viscosity. This balance causes the total energy within the system to stabilize eventually.

3. Results and Discussion

Figure. 1 illustrates the evolution of TKE over time in different Reynolds numbers, showcasing a strong concordance in the shape of the curves obtained by the literature, but not the magnitude.

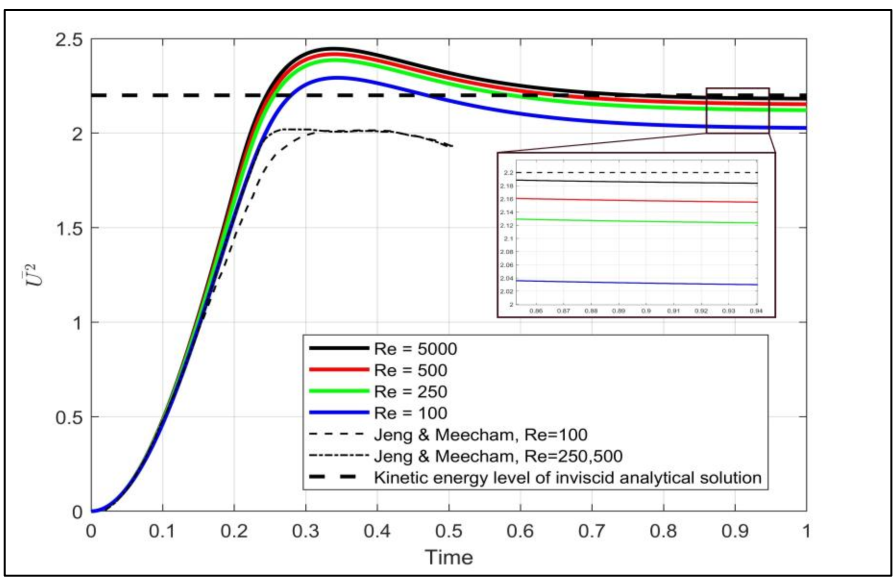

Fig. 1: The comparison between the evolution of kinetic energy over time in our simulation versus Jeng & Meecham results [3]. The horizontal dashed line shows the magnitude of TKE in the inviscid forced Burgers turbulence Eq. (3). As shown, the total kinetic energy (TKE) of the system nears the TKE of the analytical inviscid solution as the Reynolds number grows, indicating a progression toward a non-viscous state. We used $N = 24576$ Fourier modes and the temporal mesh size chosen to be $\delta t = 1e - 5$.

The difference in the magnitude of kinetic energy between our results and those of Jeng and Meecham warrants further attention and we also confirm that the values of analytical TKE closely align with the energy content of our model, Fig.1. Additionally, there is a notable alignment between the velocity field in our simulations and the inviscid analytical solution, as illustrated in Figure 2.

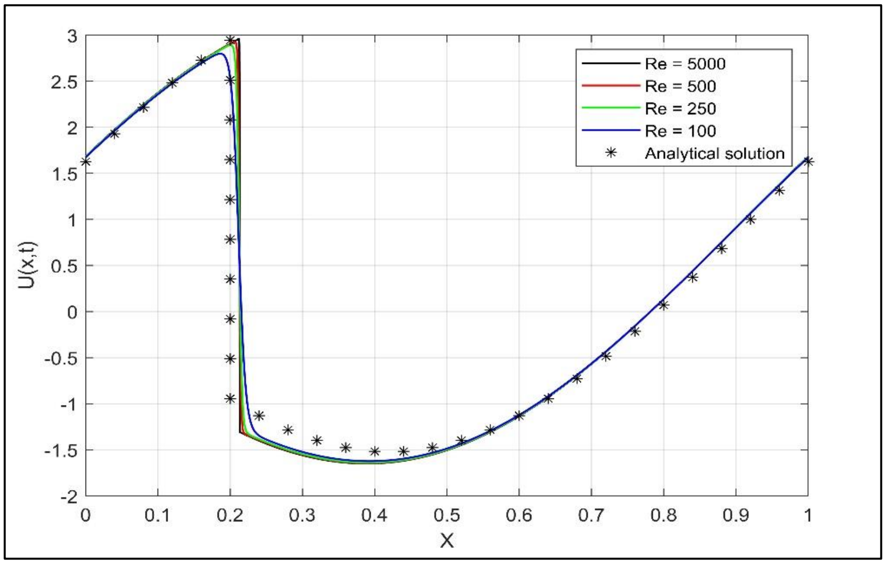

Fig. 2: The velocity profile of forced Burgers turbulence in various Reynolds numbers stay nearly identical, except within the dissipation region. No significant shift in shock position was observed, in contrast with the results obtained by Jeng and Meecham [3]

Note that this difference maybe attributed to a loss of energy, potentially resulting from inherent errors in the finite difference scheme in Jeng et al.'s model. This discrepancy is observed in three distinct ways: firstly, weaker maximum velocity values are observed at the shock position; secondly, a notable shift in shock positions is apparent in their results, details of which are more evident in Fig. 3, yet are not detected in our results. Lastly, and importantly, their model reached equilibrium earlier than expected ($t \geq 0.25$) compared to almost ($t \geq 0.8$) in our simulations.

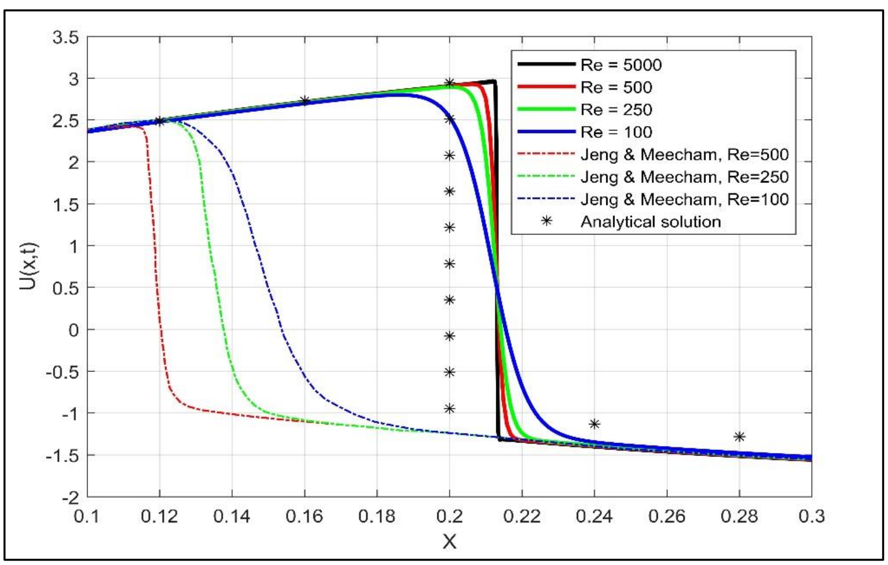

Fig. 3: The details of shock structure in different Reynolds numbers. The difference in shock position of our simulations and Jeng and Meecham [3] results are obvious

As illustrated in Fig. 4, the discrepancies in TKE level—defined as the sum of energy levels across wavenumber spectrum—are attributed to differences in energy distribution at smaller scales, where different dissipation rates where assigned based on various Reynolds numbers. It is accurate to state that once equilibrium is attained, the energy spectrum remains unchanged. However, prior to reaching this state, viscous dissipation plays a pivotal role in determining the spectrum's distribution, imparting a distinctive shape governed by scaling laws within both the inertial and dissipation subranges.

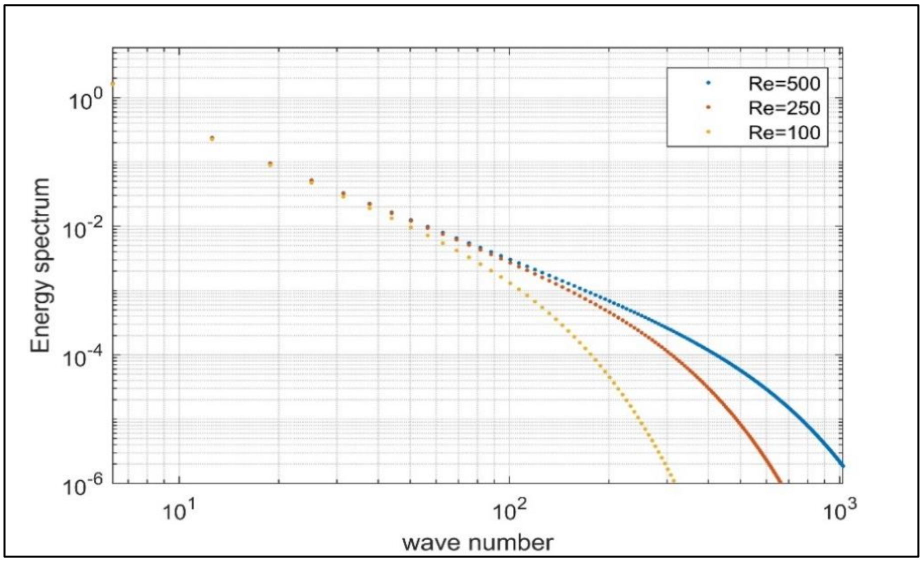

Fig. 4: The energy levels at equilibrium state across the wavenumber spectrum reveal notable differences in energy distribution that are confined to smaller scales. While these discrepancies have a minimal impact on the system's overall energy content, they are still decisive in determining the total kinetic energy level. $N = 1536$ including the de-aliased modes and $\delta t = 5e - 5$

The behaviour of the dissipation rate prior to and after reaching the equilibrium state is illustrated in Fig. 5. These effects come into play, dissipating energy, and quickly stabilizing the system as soon as the energy cascades into the dissipation range.

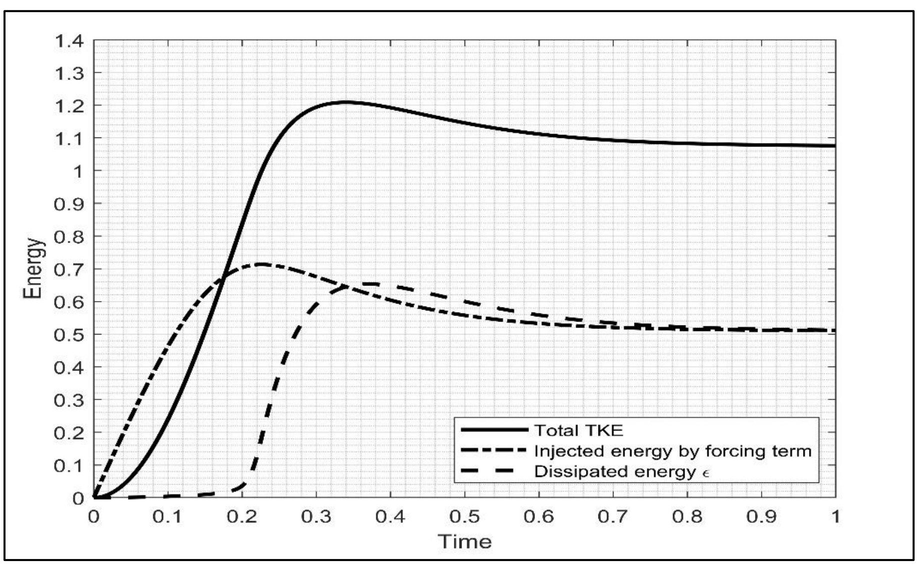

Fig. 5: Formation of Equilibrium state between injected and dissipated energy level in the forced Burgers turbulence. Total kinetic energy tends toward a constant value when the system reaches equilibrium state $t \geq 0.8$.

4. Conclusion

Based on our discussions so far, we have observed that the TKE of the system approaches the analytical TKE of the inviscid solution as the Reynolds number increases and moving towards a non-viscous condition. As such, at very high Reynolds numbers, the discrepancy between the analytical inviscid TKE and the numerical TKE, in the context of diminishing viscosity, becomes negligible. Therefore, we may conclude that the independency of the TKE from Reynolds number can be defined just in the limit of vanishing viscosity.

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