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Stagnation Point Flow of a Non-Newtonian Williamson Fluid over a Stretching/Shrinking Sheet: Existence Proofs, Dual Solutions, and a Stability Analysis

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Abstract Since non-Newtonian fluids are often encountered in engineering devices, the nonlinear boundary layer equations governing the flow and heat transfer properties of a non-Newtonian Williamson fluid over a stretching $(c > 0)$ or shrinking $(c < 0)$ sheet near the stagnation point are analyzed using two closely interrelated approaches. First, employing the shooting argument, it is proved that a unique solution exists when $c \in (-1, \infty)$ and second, using the BVP4C solver in MATLAB, two different solution branches are reported on the interval $[c_T, -1]$, where c_T is the bifurcation point. The c_T values become more negative with increasing values of the Williamson parameter λ, marking the broadening of the solution range. Furthermore, the first solution branch continues for large positive values of c, whereas the second branch seems to cease at $F''(0) = 0$ as $c \to -1$. The smallest eigenvalue computed using temporal stability analysis of these solutions is found positive for the first branch, indicating that this branch is physically stable. These findings are relevant to various industrial processes involving non-Newtonian fluids, such as polymer processing and coating applications. Finally, an asymptotic expression is derived to provide insights into the behavior of large c.

***Keywords*:** Williamson fluid, Stagnation point, Stretching or Shrinking, Existence-Uniqueness, Dual solutions, Stability analysis, Asymptotic analysis.

1. Introduction

It is common knowledge that many industrial (such as paints and coatings) and physiological (such as blood and plasma) fluids exhibit complex flow behavior that the classical Newtonian fluid model cannot adequately describe. To gain a better understanding of such fluids, numerous models (non-Newtonian) have been suggested over the years to take into account the unique characteristics of these fluids, including their viscoelastic properties, shear-thinning or shear-thickening behavior, and time-dependent responses [1, 2]. The nonlinear relationships between the stress tensor and the deformation rate tensor for non-Newtonian fluids give rise to complex equations. Undoubtedly, it is challenging to prove the existence and uniqueness/non-uniqueness of a solution to these equations and obtain their numerical solution.

This paper focuses on the robust model put forward by Williamson to describe pseudoplastic fluids [3]. A large number of published works, for example, the study of the flow of a thin layer of pseudoplastic fluid over an inclined solid surface [4], the peristaltic flow of chyme in the small intestine [5], blood flows through a tapered artery with stenosis [6], and some boundary layer flows of Williamson fluid [7], to mention a few, demonstrate the adequacy of Williamson's model in describing many frequently observed industrial and physiological fluids like polymer solutions, paints, blood, and plasma. Further, one can go through the investigations [8, 9] for Williamson fluid flows in various geometries (especially stagnation point flow and stretching/shrinking surface) under diverse physical conditions. Due to its immense engineering and industrial applications, the stagnation-point flow of a viscous or non-Newtonian fluid has been the subject of several investigations [10, 11]. Another significant aspect of boundary layer flow involves the stretching or shrinking phenomena [12].

A review of the literature suggests that the flow generated by a shrinking sheet has recently captured the interest of researchers due to its intriguing physical characteristics and growing practical implementations. Wang [11] introduced the concept of flow resulting from a shrinking sheet and showed that the solution is not unique to a particular domain. Subsequently, several research papers [13-15] have been published addressing the shrinking sheet problem. The works mentioned above were devoted to finding multiple solutions and their stability analysis. Analyzing multiple solutions and stability is crucial in engineering analysis as it enables the determination of the physical relevance of a steady-state solution. In the context of stability analysis, Merkin [16] first found that in time-dependent problems of steady-state flows, only the stable upper branch solution is physically possible, as it has the smallest positive eigenvalue. In contrast, the unstable lower branch solution is not physically relevant. Recent studies in references [17, 18] have discussed the stability of multiple solutions associated with stretching or shrinking surfaces.

In the last few decades, numerous investigations have demonstrated the mathematical proof of the existence and uniqueness of solutions in boundary layer fluid flow problems. Miklavčič and Wang [19] established the existence and uniqueness of the similarity solution for the equation describing the flow caused by a shrinking sheet with suction. Gorder et al. [20] examined the results concerning the existence and uniqueness of solutions over the interval $[0, \infty)$ for the stagnation-point flow of a hydromagnetic fluid over a stretching or shrinking sheet. Pallet et al. [10] proved the existence and uniqueness of a solution for oblique stagnation point flow by using the topological shooting argument. However, to the best of the authors' knowledge, only a limited number of articles are devoted to answering the question of the existence of a unique solution, see [21, 22] and the references therein for a detailed understanding of the methodology used.

Motivated by the investigations mentioned above and recognizing the widespread applications of problems involving stretching/shrinking sheets and non-Newtonian fluids in engineering and industries, we consider the stagnation point flow of the Williamson fluid model over a stretching/shrinking surface here. Primarily, the following research questions are addressed

- How can the existence and uniqueness of solutions for the stretching/shrinking parameter $c > -1$ be mathematically established?
- What is the critical point c_T, and how does the nature of the solution change when $c < c_T$?
- What are the characteristics of dual solutions in the shrinking parameter range $c_T \le c \le -1$?
- How can a linear stability analysis be conducted to identify stable solutions?
- What are the effects of the non-Newtonian parameter λ and shrinking parameter c (specifically $c \le -1$) on the velocity and temperature profiles in the dual solution?
- How do the expressions for shear stresses and the Nusselt number behave for large c ?

2. Governing Equations

Consider a steady, two-dimensional, incompressible flow of Williamson fluid over a horizontal linearly stretching/ shrinking sheet with no body force coincides with the plane where $y = 0$. The flow is restricted to the area where $y > 0$. The sheet's velocity is represented by $u_w(x) = p_1 x$, where $u_e(x) = bx$ (where $b > 0$) characterizes the free stream velocity. Here, the constant $p_1 > 0$ represents stretching and $p_1 < 0$ represents shrinking. Let (u, v) be the velocity component in (x, y) direction and T be the temperature. Following [7], the boundary layer equations are expressed as

$$\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\mu_0}{\rho}\left(\frac{\partial^2 u}{\partial y^2} + 2\Gamma\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{1}$$

and

$$\left(v\frac{\partial T}{\partial y} + u\frac{\partial T}{\partial x}\right) = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}. \tag{2}$$

Here, p represents pressure, c_p indicates the specific heat, k represent the thermal conductivity and Γ be time constant. Relevant boundary conditions for the stagnation point flow of Williamson fluid over stretching/shrinking sheet [7] are

$$(u, v, T) = (u_w, 0, T_w) \text{ at } y = 0, \tag{3}$$
$$(u, T) \to (u_e, T_\infty) \text{ as } y \to \infty, \tag{4}$$

where T_w and T_∞ are the surface and ambient temperature, respectively. Using the Bernoulli equation and neglecting the hydrostatic term, $-\frac{1}{\rho}\frac{\partial p}{\partial x} = u_e\frac{du_e}{dx}$, gives $-\frac{1}{\rho}\frac{\partial p}{\partial x} = b^2 x$.

Following the similarity transformations $u = bxF'(s), v = -\sqrt{b\nu}F(s), T = T_w + (T_w - T_\infty)\zeta(s)$ [7], where $s = \sqrt{\frac{b}{\nu}}y$, the equations (1)-(2) become

$$F''' - F^2 + 1 + FF'' + \lambda F''F''' = 0, \tag{5}$$
$$\zeta'' + \mathrm{Pr}F\zeta' = 0, \tag{6}$$

where $\lambda = 2\Gamma x\sqrt{\frac{b^3}{\nu}}$ be the non-Newtonian Williamson parameter and $\mathrm{Pr} = \frac{\nu\rho c_p}{k}$ is the Prandtl number. Also, the boundary conditions (3)-(4) become

$$F(0) = 0, F'(0) = c, F'(\infty) \to 1, \zeta(0) = 1, and\ \zeta(\infty) \to 0, \tag{7}$$

where $c = \frac{p_1}{b}$ represents the stretching ($c > 0$) or shrinking ($c < 0$) parameter.

3. Existence and uniqueness results for $c > -1$

3.1 Existence for $F(s)$

The existence of a solution for the boundary value problem in equations (5) and (7) is analyzed using the topological shooting method. This method entails the investigation of a corresponding group of initial value problems (IVP), denoted as the ODE (5) and (7) (except the condition at ∞), in conjunction with an additional initial condition specified as $F''(0) = a$, where a can take any arbitrary values. Then, the solution of the IVP depends on both s and a and is denoted as $F(s; a)$. Although each a yields a solution for the IVP, not all these solutions will satisfy the boundary conditions (7). Therefore, it is necessary to determine a suitable value for a that satisfies the condition at ∞. To prove the existence of a solution, the range $c > -1$ is divided into two parts: $-1 < c \leq 1$ and $c > 1$. For $c = 1$, the identity function $F(s) = s$ is a solution of (5). In this case $F''(0) = a = 0$ for all s, therefore we did not consider the case $c = 1$ in our proof.

3.1.1 Existence Proof for $-1 < c < 1$

Let us assume two sets P and Q are subsets of $(0, \infty)$, defined by

$$P = \{a > 0: \exists s_1 > 0 \text{ such that } F''(s_1; a) = 0 \text{ and } c < F'(s; a) < 1 \text{ for } s \in (0, s_1]\},$$

$$Q = \{a > 0: \exists s_1 > 0 \text{ such that } F'(s_1; a) = 1 \text{ and } 0 < F''(s; a) < a\ for\ s \in (0, s_1]\}. \tag{8}$$

Lemma 1. *P and Q are open sets with no elements in common.*

Proof: Clearly P and Q have no element in common. Let $a_1 \in P$ then $\exists\ s_1 > 0$ such that $F''(s_1; a_1) = 0$ and $c < F'(s; a_1) < 1$ for $s \in (0, s_1]$. Since $F'''(s_1; a_1) = \left(F'(s_1; a_1)\right)^2 - 1 \neq 0$, therefore, using the property of continuous functions $\exists$ a neighborhood of a_1 such that for all points in the neighborhood, $F'''(s)$ have the same sign as $F'''(s_1; a_1)$. Thus $F''(s)$ has a root with $c < F'(s) < 1$. This shows that P is an open set. Similarly, one can prove that Q is open as well.

Lemma 2. *P is non-void.*

Proof: We claim that when a is very small, it is in P. Let $a = 0$, then $F'''(0; a) < 0$ for all a. Thus, in a small enough vicinity around $s = 0$, it holds that $F''(s; 0) < 0$ and $F'(s, 0) < 1$. Then, through the continuous solutions of the IVP, along with its initial conditions, there is a positive number a for which $F''(s; a) < 0$ and $F'(s; a) < 1$ hold for all values of s in the vicinity of $s = 0$. But $F''(0; a) = a > 0$, implies $\exists$ a $\delta > 0$ such that $F''(\delta; a) = 0$ and $F'(s; a) < 1$ for $s \in (0, \delta]$. Hence for small a (> 0), it is in P.

Lemma 3. *Q is non-void.*

Proof: We claim that when a is very large, it is in Q, that is $F' = 1$ in (0,1] strictly before $F'' = 0$. If this is not the case, then the following possibilities must occur : (i) $F''(s;a) = 0$ for some point in (0,1] for which $F'(s;a) < 1$, (ii) $F''(s;a) > 0$ and $F'(s;a) < 1$ in (0,1], and (iii) $F''(s;a) = 0$ and $F'(s;a) = 1$ occur concurrently. If possible, let $\exists\, c_1 \in (0,1]$ such that $F''(c_1;a) = 0$ with $c_1 < F'(s;a) < 1$ for $s \in (0, c_1]$. By integrating, we get $c_1 s < F(s;a) < s$. Now let $\bar{F} = \int_0^s \frac{F}{1+\lambda F''} dt$ and integrating (5) from 0 to s, we get

$$F''(s)e^{\bar{F}(s)} - F''(0)e^{\bar{F}(0)} = \int_0^s \frac{1-F'^2}{1+\lambda F''} dt, \tag{9}$$

$$\Longrightarrow F''(s)e^{\bar{F}(s)} = a + \int_0^s \frac{1-F^2}{1+\lambda F''} dt. \tag{10}$$

Let $H = \frac{1-F'^2}{1+\lambda F''} dt > 0$, then form (10) we have

$$F''(s)e^{\bar{F}(s)} = a + Hs. \tag{11}$$

Then for $s \in (0, c_1]$

$$F''(s) \geq (a+H)e^{-\bar{F}(s)}. \tag{12}$$

Thus, for large $a, F''(s;a) > 0$ for all s, leading to a contradiction. Similarly, it can be shown that the second statement cannot occur for sufficiently large values of a. If the third case occurs, then from (5), we get $F'''(s;a) = 0$. That implies that $F'(s) = 1$, which contradicts the fact that $F'(0) = c \neq 1$. Therefore, sufficiently large a belongs to Q.

Theorem ***1***. *For any $\lambda \geq 0$, equations (5) and (7) have a solution. Also, the solution is monotone in nature.*

Proof: As $(0,\infty)$ is a connected set, and both P and Q are non-empty, open, and disjoint from each other, it follows from the definition of a connected set that the union of P and Q cannot be equal to $(0,\infty)$. Therefore $\exists\, l > 0$ such that $l \notin P$ and $l \notin Q$. Also, Lemma 3 implies that $F''(s;l) = 0$ and $F'(s;l) = 1$ do not occur simultaneously. Consequently, there is only one possibility that $F''(s;l) > 0$ and $c \leq F'(s;l) < 1\ \forall s$. Now, from equation (5), it is observed that as $F'(\infty;l)$ approaches 1, implies the existence of a monotonically increasing solution to the boundary value problem (5), (7).

For the case $c > 1$, the proof follows similarly by defining the sets U and V are subsets of $(-\infty, 0)$,

$$U = \{a < 0\colon \exists\, s_1{}' > 0 \textit{ such that } F''(s_1{}';a) = 0 \textit{ and } 1 < F'(s;a) < c \textit{ for } s \in (0, s_1{}']\},$$

$$V = \{a < 0\colon \exists\, s_1{}' > 0 \textit{ such that } F'(s_1{}';a) = 1 \textit{ and } c < F''(s;a) < 0 \textit{ for } s \in (0, s_1{}']\}.$$

3.1.2 Uniqueness Proof for $-1 < c \leq 1$

Theorem 2. For any $\lambda \geq 0$, the solution is unique.

Proof: We will prove this theorem by using the method of contradiction. Let us assume that $\exists\, a_1, a_2$ (values of $F''(0)$) such that $F(s;a_1)$ and $F(s;a_2)$ are the corresponding solutions. Apply MVT on the function F' in the interval $[a_1, a_2]$ and as $s \to \infty$ then $\exists\, a^* \in [a_1, a_2]$ such that $\frac{\partial F'}{\partial a}(\infty, a^*) = 0$. Next, let $\frac{\partial F'}{\partial a} = w'(s;a)$ and differentiating (5) and using the boundary conditions (7), we have

$$w''' - 2F'w' + \lambda(w'''F'' + F'''w'') + (F'w' + Fw'') = 0, \tag{13}$$

with

$$w(0) = 0, w'(0) = 0, w''(0) = 1, w'''(0) = \frac{-\lambda F'''(0)}{1+\lambda a}. \tag{14}$$

Further differentiating (13), we have

$$w^{iv}(1+\lambda F'')+2\lambda F'''w'''+\lambda w''F^{iv}-F''w'+Fw''=0. \tag{15}$$

Now, from (14), we can say that $\exists\, s_1>0$ such that $w'(s;a)>0, w''(s;a)>0, w'''(s;a)<0$ for $s<s_1$. Specifically, the function $w'(s;a)$ is convex downwards, initially increasing, and it has a maximum value to reach zero. Let the maximum value occur at s_2. Consequently, $w'''(s_2;a)=0$ and $w^{iv}(s;a)\le 0$ for $s<s_2$. Also, $w^{iv}(s_2)=0$. But equation (15) implies

$$w^{iv}(s_2)=\frac{1}{1+\lambda F''(s_2)}\left(-2\lambda w'''(s_2)F'''(s_2)+F''(s_2)w'(s_2)\right)>0, \tag{16}$$

a contradiction. However, up until the point $s_2, w(s;a)$ and all its derivatives up to $w'''(s;a)$ are growing positively. Hence, $F(s;a)$ and all its derivatives up to $F'''(s;a)$ are increasing functions. Therefore, for any a in the interval $[a_1,a_2]$, $w'(s_2,a)\neq 0$ which contradicts the MVT of F'. Hence, the proof is complete.

For the case c >1, similar to Theorem 2, it can be shown that the solution is unique.

3.2 Existence for $\boldsymbol{\zeta(s)}$

Theorem 3. If $\zeta(s)$ is a twice differentiable function satisfying (6) with boundary condition (7), then $\zeta(s)$ is of the form

$$\zeta(s)=\frac{\int_s^\infty\left(e^{-\int_0^s PrFds}\right)ds}{\int_0^\infty\left(e^{-\int_0^s PrFds}\right)ds}. \tag{17}$$

4. Numerical Results and Discussion

To validate our results, we compare the values of $F''(0)$ (when non-Newtonian parameter $\lambda=0$) on the stretching/shrinking sheet with Ishak et al. [13] (see Table 1). An increase in |c| leads to a decrease in the values of $F''(0)$ in the first solution, while it has the opposite effect in the second solution. In Table 1, $F''(0)$ gives two different values for some selected negative values of c, but after crossing the point -1, it provides only a single value. The point c_T connects both solution branches, and when $c\ \to -1$, no such critical point exists, and after crossing the point -1, it becomes a single branch. Our theoretical results are also closely connected with the above fact as $c\to -1, F''(0)\ge 0$. If $F''(0)>0$, then from (5), it is found that $F'''(0)=0$. Consequently, $F''(0)=0$, and all subsequent derivatives are zero at $s=0$, which cannot satisfy the conditions $F'(0)=-1$ and $F'(\infty)\to 1$. Therefore, a unique solution exists when $c>-1$, and dual solutions occur for $c_T\le c\le -1$, and there is no solution for $c<c_T$. The critical point c_T for $\lambda=0.1$ and 0.3 are -1.24701 and -1.24768 (see Figs. 1-2). The solution domain expands with increasing λ, and c_T is more negative for the non-Newtonian case than the Newtonian case, highlighting that λ plays a significant role in the existence of solutions, as supported by theoretical results. Fig. 3 demonstrates a significant decrease in the velocity profile $F'(s)$ with increasing λ for both solution branches. It is observed that the thickness of the momentum boundary layer is larger for Newtonian fluid than for non-Newtonian fluid. The temperature profile for both solutions increases with the non-Newtonian parameter λ (see Fig. 4), resulting in a rise in the thickness of the thermal boundary layer. Fig. 5 shows that $F'(s)$ decreases in the first solution but increases in the second solution as $|c|$ increases. Conversely, $\zeta(s)$ increases with $|c|$ in the first solution while decreasing in the second solution (see Fig. 6). The momentum and thermal boundary layer thicknesses are found to be smaller in the first solution compared to the second solution. In Fig. 7, $F(s)$ decreases in the first solution but increases in the second solution as $|c|$ increases. Initially, each curve shows a decline, reaching certain negative values for small s. However, these values gradually increase and become positive beyond a certain distance from the sheet. In Table 2, the smallest eigenvalues for both solutions are computed numerically for different shrinking parameters $|c|$. In the first solution branch, the eigenvalues are observed to be real and positive (indicating a stable solution), while in the second solution branch, they are negative (indicating an unstable solution).

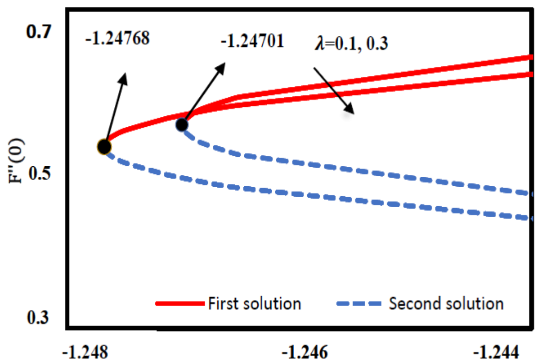

Fig. 1: Effect of λ on $F''(0)$.

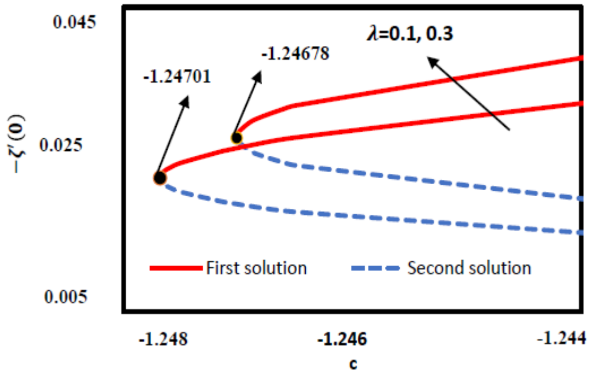

Fig. 2: Effect of λ on $-\zeta'(0)$.

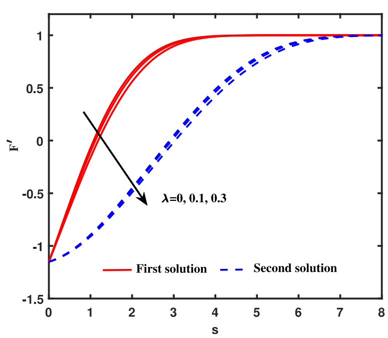

Fig. 3: Effect of λ on $F'(s)$.

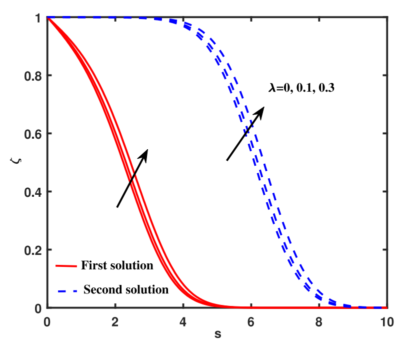

Fig. 4: Effect of λ on $\zeta(s)$.

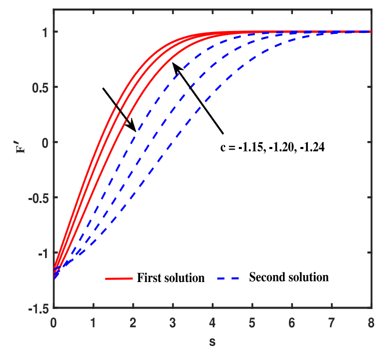

Fig. 5: Effect of c on $F'(s)$.

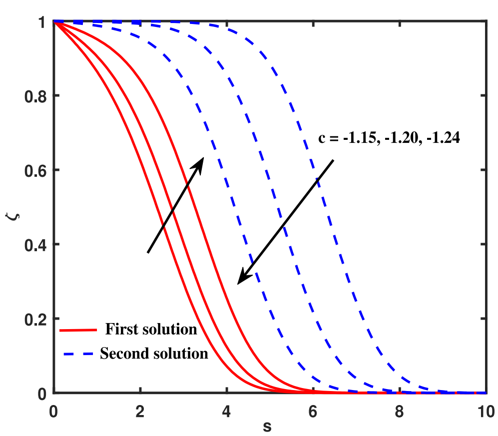

Fig. 6: Effect of c on $\zeta(s)$.

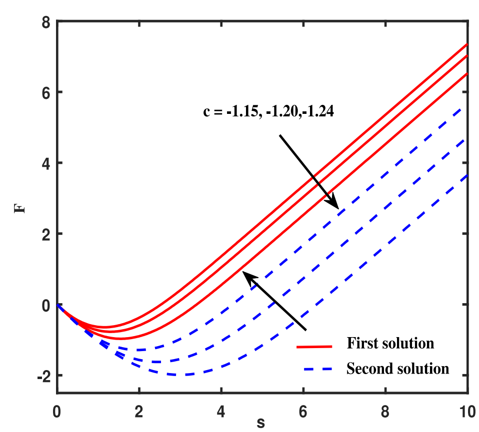

Fig. 7: Effect of c on $F(s)$.

Table 1: Comparison of $F''(0)$ for various values of λ and c.

λ	c	Present		Ishak [13]	
		First Solution	Second Solution	First Solution	Second Solution
0	-0.25	1.402240	-	1.402241	-
	-0.50	1.495669	-	1.495670	-
	-0.75	1.489298	-	1.489298	-
	-1	1.328816	0	1.328817	0
	-1.15	1.082231	0.116701	1.082231	0.116702
	-1.20	0.932473	0.233649	0.932474	0.233650
	-1.2465	0.584291	0.554281	0.584295	0.554283
0.3	-0.25	1.254506	-	-	-
	-0.75	1.321493	-	-	-
	-0.9	1.262148	-	-	-
	-1	1.187971	0	-	-
	-1.12	1.036961	0.064495	-	-
	-1.18	0.911318	0.163578	-	-
	-1.22	0.778358	0.283952	-	-
	-1.24765	0.536212	0.519569	-	-
	-1.24768	0.528127	0.528547	-	-

Table 2: Smallest eigenvalues for different λ.

λ	c	First solution	Second solution
0.1	-1.24	0.157272	-0.258123
	-1.19	0.573241	-0.598794
	-1.18	0.627739	-0.638914
0.3	-1.24	0.016590	-0.348644
	-1.21	0.341042	-0.571240
	-1.20	0.405736	-0.618171

Table 3: Asymptotic values of $c^{-3/2}F''(0)$ and $c^{-1/2}\zeta'(0)$ for $\lambda = 0.5\, c^{-3/2}$.

c	$F''(0)$	$\zeta'(0)$	$c^{-3/2}F''(0)$	$c^{-1/2}\zeta'(0)$
5	-12.984637	-1.359882	-1.161381	-0.608202
20	-115.56896	-2.542579	-1.292100	-0.568538
60	-608.62809	-4.342031	-1.309559	-0.560554
100	-1312.3680	-5.590453	-1.312368	-0.559045
200	-3117.4350	-7.890453	-1.314312	-0.557939
∞	-	-	-1.316134	-0.556919

4 Conclusions

The research delved into the boundary layer stagnation-point flow and convective heat transfer on a linearly stretching/shrinking surface in non-Newtonian Williamson fluid. The main findings of this study can be outlined as follows

- The existence of a unique solution to the nonlinear equation is proved for stretching/shrinking parameter $c \in (-1, \infty)$. Dual solutions exist for c $\in [c_T, -1]$, and there does not exist any solutions for $c \in (-\infty, c_T)$.

- The velocity profile $F'(s)$ decreases with non-Newtonian parameter λ in both solution branches, whereas the temperature profile $\zeta(s)$ increases with λ. In the first solution branch, the boundary layer thickness (for both momentum and thermal) is smaller compared to the second solution branch. Additionally, the solution domain expands with increasing λ.

- Stability analysis indicates that the first solution branch is physically acceptable, as all the smallest eigenvalues are positive, whereas the second solution branch is unstable.

- An asymptotic solution for large $c > 0$ shows that the expressions $F''(0) \sim -1.316134\, c^{3/2}$ and $\zeta'(0) \sim -0.556919$ $c^{1/2}$ as $c \to \infty$.

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