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Transient Slow Translation of a Composite Sphere in Unbounded fluid

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***Abstract* -** The transient translation of a composite spherical particle, consisting of an impermeable hard sphere core and a permeable porous surface layer, in a viscous fluid at low Reynolds number induced by suddenly applied continuous force and are analytically studied. Transient translation is of great interest to researchers in various scientific fields and industrial processes, such as centrifugation, sedimentation, filtration, flocculation, electrophoresis, microfluidics, aerosol technology, and suspension rheology. By solving the unsteady Stokes and Brinkman equations that respectively govern the fluid flows outside and inside the porous surface layer through the Laplace transform, closed-form formulas of the starting linear velocity of the composite sphere as functions of the relevant parameters are obtained. The velocity increases over time from initial values of zero to their final values, while the linear acceleration of the composite sphere decreases over time, eventually approaching zero. At any elapsed time, the velocity and acceleration increase monotonically and significantly with increasing relative spatial volume and fluid permeability of the porous surface layer of the composite sphere. The transient linear velocity of the composite sphere is generally increasing functions of the porosity of the surface layer, but may decrease slightly with increasing porosity when the particle-to-fluid density ratio is small. The linear acceleration increases with the increase of surface layer porosity in the early stage, no longer change monotonically with its increase in the middle stage, and decrease with its increase in the later stage. The transient linear velocity of the composite sphere decreases with the increase of its relative density, but the linear acceleration only decreases with the increase of the relative density in the early stage and increase with the increase of the relative density in the later stage.

***Keywords*:** Transient translation; Creeping flow; Composite sphere; Hydrodynamic drag force; starting linear and angular velocities

1. Introduction

The translation of colloidal particles in viscous fluids at extremely small Reynolds numbers are of great interest to researchers in various scientific fields and industrial processes, such as centrifugation, sedimentation, filtration, flocculation, electrophoresis, microfluidics, aerosol technology, and suspension rheology. The analytical research in this subject originated from Stokes' classic work on the steady-state motion of non-slip hard spheres in an incompressible Newtonian fluid[1,2] and was extended to the motions of slip hard spheres, fluid spheres, and porous spheres.

A composite spherical particle of radius b is an impermeable hard sphere core of radius a coated with a porous surface layer of thickness $b-a$, which is permeable to fluids. In the limits $a=b$ and $a=0$, the composite particle reduces to entirely hard and porous spheres, respectively. A biological cell with protein surface attachments and a polystyrene latex with macromolecule surface appendages are examples of a composite particle. To achieve steric stabilization of a colloidal suspension, polymeric chains are purposely adsorbed on a hard particle to form a porous layer. The translation of composite spheres were extensively analyzed for decades. A convenient approach which adds a second-order viscous term to the Darcy equation or a zero-order friction term to the Stokes equation for fluid flows within a porous medium was founded by Brinkman . Chen and Ye calculated the drag force, and stresslet for the motions of a composite sphere in a simple shear flow by using the Brinkman equation for the fluid flow in the porous surface layer and the Stokes equation for the flow external to the particle with suitable boundary conditions on the inner and outer surfaces of the porous surface layer.

The linear velocity U_∞ of a composite sphere of radius b translating under the applied force F_A along/about its diameter in a Newtonian fluid of viscosity η in the steady state of low Reynolds numbers have been obtained by solving for the Brinkman and Stokes equations, with the result[3,4]

$$U_\infty = \frac{\lambda F_A}{6\pi\eta}\{(\lambda a \sinh\lambda b - \cosh\lambda a)[(W+3\lambda b)D_1 + 3(\lambda^2 a^2 - 1)D_2 - 6\lambda a]\}$$
$$\times\{W\lambda a\cosh\lambda a - 3\lambda^2 a^2(R+\lambda a\sinh\lambda a) + [(R\lambda a - \lambda b\cosh\lambda a)W$$
$$+3\lambda^3 a^2 b\sinh\lambda a]D_1 + [W\cosh\lambda a + 3\lambda^2 a^2(R\lambda a - \sinh\lambda a)]D_2\}^{-1}, \quad (1)$$

where

$$D_1 = \cosh(\lambda b - \lambda a), \quad (2a)$$

$$D_2 = \sinh(\lambda b - \lambda a), \quad (2b)$$

$$R = \lambda b\sinh\lambda b - \cosh\lambda b, \quad (3a)$$

$$W = 2\lambda^3 b^3 + \lambda^3 a^3 + 3\lambda a, \quad (3b)$$

a is the radius of the hard core, and λ is the reciprocal of the square root of the fluid permeability or flow penetration length in the porous surface layer. Note that the translation of the spherical particle is not coupled with each other; viz, U_∞ are not related to F_A, respectively. In the situations $a=b$ and $a=0$, Eqs. (1) degenerate to the Stokes results ($U_\infty = F_A / 6\pi\eta b$) for a hard sphere and corresponding results for an entirely porous sphere (which usually models a permeable polymer coil or floc of nanoparticles), respectively, of radius b. In the limiting cases of $\lambda b = 0$ (fully permeable or no resistance to the fluid flow in the porous layer of the composite sphere) and $\lambda b \to \infty$ (impermeable or no relative fluid flow in the porous layer), Eqs. (1) also reduce to the Stokes results for a hard sphere of radii a and b, respectively.

Although the basic formulas for the slow motion of colloidal particles were established primarily in the steady state, their transient behavior is also important for assessing the validity of the steady-state assumption. The time evolution of particles' linear is related to various particle dynamics within milliseconds. The low Reynolds number responses of the hydrodynamic force exerted on a hard particle, a porous sphere, and a composite sphere to unsteady particle translation or to unsteady viscous fluid flow have been examined. On the other hand, transient translation of hard and porous [5] spheres induced by suddenly applied force were also studied. However, the start-up transient translational motions of composite particles have not yet been investigated. In this paper, the starting translation of a composite sphere due to the sudden application of continuous force along/about its diameter are analytically studied. Explicit formulas for the transient linear of the composite sphere in the Laplace transform are obtained.

2. Analysis

we consider the axisymmetric creeping flow activated by the starting migration of a composite sphere of radius b, composed of a hard sphere core of radius a and a porous surface layer of thickness $b-a$, in a viscous fluid in the spherical coordinates (r,θ,ϕ) with origin at the center of the composite particle, as shown in Fig. 1. A constant force $F_A\mathbf{e}_z$ is applied to the initially stationary particle at time $t=0$ and lasts afterward, where $\mathbf{e}_z$ is the unit vector along the z axis (with $\theta=0$). The resulting transient migration velocity $U(t)\mathbf{e}_z$ of the particle (without rotation) to be determined increases from zero at $t=0$ to its terminal value U_∞ given by Eq. (1) as $t\to\infty$.

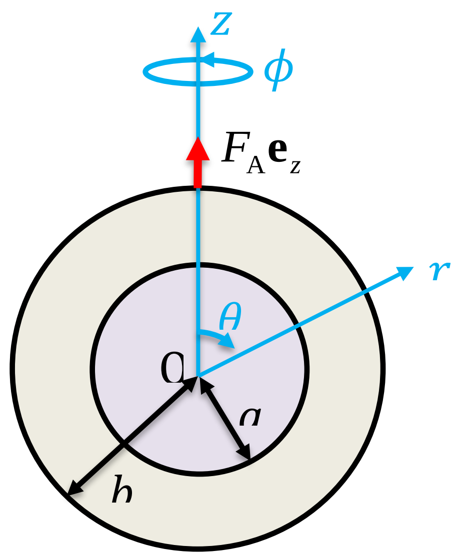

Figure 1. Geometrical sketch for a composite sphere undergoing translation .

The fluid velocity $\mathbf{v}(r,\theta,t)$ and hydrodynamic pressure $p(r,\theta,t)$ are governed by the Stokes and Brinkman equations in the transient state[5]:

$$[1-h(r)(1-\varepsilon)]\rho\frac{\partial\mathbf{v}}{\partial t}=\eta\nabla^2\mathbf{v}-\nabla p-h(r)f(\mathbf{v}-U\mathbf{e}_z), \tag{4}$$

where η and ρ are the fluid viscosity and density, respectively, f and ε are the hydrodynamic resistance factor and porosity, respectively, of the surface layer of the composite sphere, and $h(r)$ is a step function equal to unity if $a\le r\le b$ and naught otherwise. Note that, inside the surface layer, $\mathbf{V}$ is the local superficial velocity over a fluid-plus-solid spatial region, whose linear dimension is much larger than the pore sizes but smaller than the layer thickness, the fluid viscosity is assumed to equal the viscosity η of the fluid outside the surface layer, and the last term in Eq. (4) involves the friction force between the fluid flow and the rigid skeleton.

Taking the curl of Eq. (4) and employing the Stokes stream function Ψ which satisfies the continuity equation $\nabla\cdot\mathbf{v}=0$ immediately [$\mathbf{v}=(r\sin\theta)^{-1}\mathbf{e}_\phi\times\nabla\Psi=-\nabla\times(\mathbf{e}_\phi\Psi/r\sin\theta)$, where $\mathbf{e}_\phi$ is the unit vector in the ϕ direction], one obtains

$$E^2\{E^2-[1-h(r)(1-\varepsilon)]\frac{1}{\nu}\frac{\partial}{\partial t}-h(r)\lambda^2\}\Psi=0, \tag{5}$$

where $\lambda^{-1}=(\eta/f)^{1/2}$ is the flow permeation length into the porous surface layer, $\nu=\eta/\rho$ is the kinematic viscosity, and E^2 is the axisymmetric Stokes operator defined as

$$E^2=\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}). \tag{6}$$

The boundary and initial conditions for the fluid flow field are

$r=a$: $\mathbf{v}=U\mathbf{e}_z$, (7)

$r=b$: $\mathbf{V}$ and $\boldsymbol{\tau}-p\mathbf{I}$ are continuous, (8)

$r\to\infty$: $\mathbf{v}=\mathbf{0}$, (9)

$t=0$: $\mathbf{v}=\mathbf{0}$, (10)

where $\boldsymbol{\tau}$ is the deviatoric stress tensor and $\mathbf{I}$ is the unit tensor.

The general solution of the stream function to Eq. (5) after applying Eq. (10) and the Laplace transform (with an overbar to the variable) is

$$\bar{\Psi}=\bar{U}\frac{b^3}{r}[C_1\frac{r^3}{b^3}+C_2+C_3\alpha(Br)+C_4\beta(Br)]\sin^2\theta \quad \text{if } a\le r\le b, \tag{11a}$$

$$\bar{\Psi}=\bar{U}\frac{b^3}{r}[C_5\frac{r^3}{b^3}+C_6+C_7\gamma(Ar)+C_8\gamma(-Ar)]\sin^2\theta \quad \text{if } r\ge b, \tag{11b}$$

where

$$\alpha(x)=x\cosh x-\sinh x, \tag{12a}$$

$$\beta(x)=x\sinh x-\cosh x, \tag{12b}$$

$$\gamma(x)=(1-x)e^x, \tag{12c}$$

$A=(s/\nu)^{1/2}$, $B=(\lambda^2+\varepsilon s/\nu)^{1/2}$, and s is the transform parameter.

The unknowns $C_1 \sim C_8$ in Eq. (11) are functions of s determined using Eqs. (7)-(9).

The substitution of Eq. (11a) into the formula $\mathbf{v}=(r\sin\theta)^{-1}\mathbf{e}_\phi\times\nabla\Psi$ and Eq. (4) leads to the Laplace transforms of the nonzero velocity components v_r and v_θ as well as pressure p of the internal fluid (with $a\le r\le b$):

$$\bar{v}_r=-V(r)\cos\theta, \tag{13a}$$

$$\bar{v}_\theta=\frac{1}{2r}\frac{\partial}{\partial r}[r^2V(r)]\sin\theta, \tag{13b}$$

$$\bar{p}=\eta\bar{U}\{\lambda^2+B^2[2C_1-C_2(\frac{b}{r})^3]\}r\cos\theta, \tag{13c}$$

where

$$V(r)=2\bar{U}\frac{b^3}{r^3}\{C_1\frac{r^3}{b^3}+C_2+C_3[Br\cosh(Br)-\sinh(Br)]+C_4[Br\sinh(Br)-\cosh(Br)]\}. \tag{14}$$

The drag force (in the z direction) exerted by the fluid on the composite particle is negative (with magnitude increasing with the elapsed time from zero at $t=0$ to F_A as $t\to\infty$) and given by

$$F_\mathrm{h}=2\pi\int_0^\pi\int_a^b\mathbf{e}_z\cdot f(\mathbf{v}-U\mathbf{e}_z)\,r^2\sin\theta\mathrm{d}r\mathrm{d}\theta+2\pi a^2\int_0^\pi[(\tau_{rr}-p)\cos\theta-\tau_{r\theta}\sin\theta]_{r=a}\sin\theta\mathrm{d}\theta, \tag{15}$$

where $\tau_{rr}=2\eta\partial v_r/\partial r$ and $\tau_{r\theta}=\eta r\partial(v_\theta/r)/\partial r$ are the nontrivial normal and tangential components, respectively, of the stress tensor $\boldsymbol{\tau}$. The substitution of Eq. (13) together with integration constants and (14) into the Laplace transform of Eq. (15) results in

$$\bar{F}_\mathrm{h}=-\frac{2}{3}\pi\eta b\bar{U}L, \tag{16}$$

where

$$\begin{aligned}L=\{&[-12B^3a(1+Ab)(a^2M_3-\lambda^2b^4)\\&+6B^{-1}\lambda^2\{(1+Ab)[a(B^4b^4+J_4)-3a(1+Aa)M_3-(b^3-a^3)B^2M_3]+bJ_3+a^{-1}b^2J_2\}\\&+\{B^3a^2bJ_3-6B^{-1}\lambda^2b[(1+Ab)J_4+J_3+J_2]+2Ba^3J_4(B^2-\lambda^2)\\&\quad+18B^{-1}\lambda^2aM_3(1+Ab)^2+Bb(9J_3+9J_2+2\lambda^2b^2J_4)\}G_1\\&+\{3B^4a^2b^2(6+3Ab-A^3b^3+J_1)+2\lambda^2[B^{-2}J_4(3+3Ab+AB^2b^3)-3b^2M_3{}^2]\end{aligned}$$

$$+6B^{-2}a^{-1}b(B^2-\lambda^2)J_2+3a[b\{A^2M_2(1+Ab)-B^2(3-A^3b^3-AbJ_1-2J_3)\}$$
$$-2B^{-2}\lambda^2\{3M_3(1+Ab)(A+B^2b)+B^2bJ_3\}]$$
$$+a^3b^{-1}[2(B^2-\lambda^2)(AbJ_4-3M_3{}^2)-B^4b^2(3-A^3b^3-AbJ_1)]\}G_2]Z+2\lambda^2b^2\}\,. \qquad (17)$$

The sum of the imposed and drag forces on the composite sphere equals its acceleration multiplied by its mass,

$$F_{\mathrm{A}}+F_{\mathrm{h}}=\frac{4}{3}\pi[a^3\rho_{\mathrm{c}}+(b^3-a^3)(1-\varepsilon)\rho_{\mathrm{p}}]\frac{\mathrm{d}U}{\mathrm{d}t}\,, \qquad (18)$$

where ρ_{c} and ρ_{p} are the densities of the solid sphere core and porous surface layer, respectively, of the composite sphere. Substituting Eq. (16) into the Laplace transform of Eq. (18), we obtain the transform of the transient migration velocity of the particle as

$$\bar{U}=\frac{3F_A}{4\pi\rho b^3s^2}[\frac{\nu L}{2b^2s}+(\frac{a}{b})^3\frac{\rho_c}{\rho}+(1-\frac{a^3}{b^3})(1-\varepsilon)\frac{\rho_p}{\rho}]^{-1}. \qquad (19)$$

3. Results and Discussion

For the sake of brevity and without loss of generality, all results will be calculated for the composite spherical particle with a porous surface layer density ρ_{p} equal to the hard sphere core density ρ_{c}. The dimensionless starting velocity $6\pi\eta bU/F_{\mathrm{A}}$ of the composite sphere computed from Eq. (19) via numerical inverse Laplace transform is plotted against the scaled elapsed time $\nu t/b^2$, core-to-particle radius ratio a/b, surface layer porosity ε, ratio of particle radius to surface layer permeation length λb, and particle-to-fluid density ratio ρ_{c}/ρ in Figs. 2-3.

For fixed values of $\nu t/b^2$, ε, λb, and ρ_{c}/ρ, as shown in Figs. 2, the dimensionless velocity $6\pi\eta bU/F_{\mathrm{A}}$ of the composite sphere decreases monotonically and significantly with increasing radius ratio a/b (i.e., a decrease in the relative volume of the porous surface layer of the composite sphere), as expected.

For specified values of $\nu t/b^2$, a/b, λb, and ρ_{c}/ρ, the dimensionless velocity $6\pi\eta bU/F_{\mathrm{A}}$ of a composite sphere generally rises with increasing surface layer porosity ε from a constant at $\varepsilon=0$ to a greater one as $\varepsilon\to 1$, as illustrated in Figs. 3 and showing that the transient velocity growth of a composite sphere with lower surface layer porosity lags behind that of a composite sphere with higher surface layer porosity. However, when the value of ρ_{c}/ρ is smaller than about 0.5, $6\pi\eta bU/F_{\mathrm{A}}$ may decrease slightly as ε increases.

The dimensionless acceleration $(6\pi\rho b^3/F_{\mathrm{A}})\mathrm{d}U/\mathrm{d}t$ of a composite sphere starting to migrate under an applied force is plotted against the scaled elapsed time $\nu t/b^2$ for different values of the core-to-particle

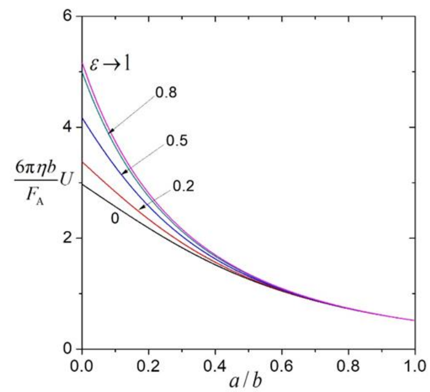

Figure 2 Dimensionless velocity $6\pi\eta bU/F_{\mathrm{A}}$ of a composite sphere at $\nu t/b^2=1$ versus the core-to-particle radius ratio a/b; $\rho_{\mathrm{c}}/\rho=1$ and $\lambda b=1$.

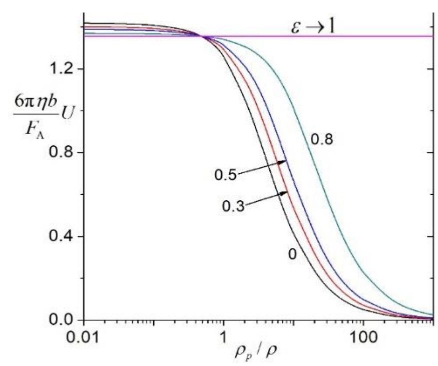

Figure 3 Dimensionless velocity $6\pi\eta bU/F_{\mathrm{A}}$ of a composite sphere at $\nu t/b^2=1$ versus the particle-to-fluid density ratio ρ_{c}/ρ; $a/b=0.5$ and $\lambda b=1$.

radius ratio a/b, porosity ε, density ratio ρ_c/ρ, and Brinkman resistance parameter λb. This acceleration monotonically decreases with increasing $\nu t/b^2$ from constant values at $\nu t/b^2=0$ and disappears as $\nu t/b^2\to\infty$. For specified values of $\nu t/b^2$, a/b, ε, and ρ_c/ρ, this acceleration (like $6\pi\eta bU/F_A$) decreases with increasing λb, as shown in Fig. 4,.

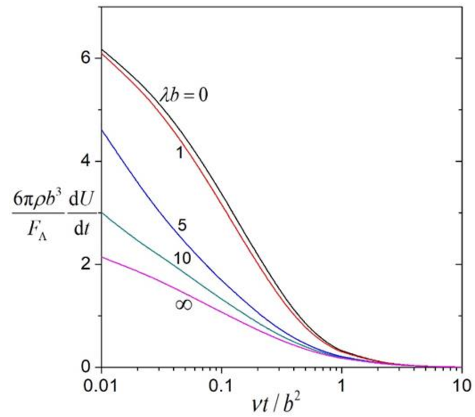

Figure 4 Dimensionless acceleration of a composite sphere $(6\pi\rho b^3/F_A)dU/dt$ versus the dimensionless time $\nu t/b^2$; $a/b=0.5$, $\varepsilon=0.5$, and $\rho_c/\rho=1$

4. Conclusion

The transient slow translation of a composite sphere of radius b, consisting of an impermeable hard sphere core of radius a and a permeable porous surface layer of thickness $b-a$, in an incompressible Newtonian fluid with density ρ, viscosity η, and kinematic viscosity ν induced by suddenly applied constant force F_A is analytically studied. By solving the unsteady Stokes and Brinkman equations that respectively govern the fluid flows outside and inside the porous surface layer through the Laplace transform, closed-form formulas of the starting linear and angular velocities of the composite sphere as functions of the scaled elapsed time $\nu t/b^2$, core-to-particle radius ratio a/b, surface layer porosity ε, ratio of particle radius to surface layer permeation length λb, and particle-to-fluid density ratios ρ_c/ρ and ρ_p/ρ are obtained. The translation of the spherical particle is not coupled with each other; viz, the linear velocity U is not related to F_A, respectively.

The dimensionless linear velocity $6\pi\eta bU/F_A$ of a composite sphere increases with the scaled time $\nu t/b^2$ from initial values of zero to their respective final values, while the dimensionless linear acceleration $(6\pi\rho b^3/F_A)dU/dt$ decreases with $\nu t/b^2$, eventually approaching zero. These transient velocities decrease monotonically and significantly with increasing radius ratio a/b and Brinkman resistance parameter λb. The linear velocities $6\pi\eta bU/F_A$ are generally increasing functions of the porosity ε, but may decrease slightly with increasing ε when the density ratios ρ_c/ρ and ρ_p/ρ are small. The linear accelerations $(6\pi\rho b^3/F_A)dU/dt$ increase with the increase of ε in the early stage, no longer change monotonically with its increase in the middle stage, and decrease with its increase in the later stage. The linear velocities $6\pi\eta bU/F_A$ and decrease with the increase of the density ratios ρ_c/ρ and ρ_p/ρ, but the linear accelerations $(6\pi\rho b^3/F_A)dU/dt$ only decrease with the increase of ρ_c/ρ and ρ_p/ρ in the early stage and increase with the increase of ρ_c/ρ and ρ_p/ρ in the later stage.

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