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Solving the One Dimensional Advection Diffusion Equation Using Mixed Discrete Least Squares Meshless Method

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Abstract– In the recent years, meshless methods are applied for solution of partial differential equations in the various engineering fields. In these types of solution procedures, nodes are the main subject for the governing equations discretization and these nodes were disturbuted in the domain in both regular and irregular scheme. High flexibility and excellent accuracy of meshless methods are two major advantages of these methods in compare with mesh based Eulerian approaches. Recently, the mixed formulation technique is used in Discrete Least square meshless (DLSM) method for the solution of elliptic partial differential equations. This method so called MDLSM is based on minimizing the least squares functional which calculated at nodes in the study area and its boundaries. The least square functional is defined as the weighted summation of the squared residuals. To construct the meshless shape functions a Moving Least Squares (MLS) approximation is used. Theoretically, the accuracy of numerical methods is improved one order by replacing the mixed formulation instead of the irreducible formulation. This theorem was approved by the obtained results for elliptic partial differential equations. In the current study, MDLSM method is developed to solve the one dimensional advection- diffusion equation as a parabolic partial differential equation. The efficiency and accuracy of proposed scheme is evaluated by a number of standard numerical examples. Results obtained through proposed scheme are then compared with analytical solutions. From this comparison, it can be concluded that the proposed method of MDLSM perform more efficient and accurate.

Keywords**:** Meshless, Mixed Discrete Least Squares, advection diffusion, Moving Least Squares

1. Introduction

Numerical methods were widely used for solving the partial differential equations (PDEs). Mesh-based numerical methods such as finite element method and finite volume method have shown their efficiency in solving the PDEs over the past few decades. However, these methods suffer from the mesh deformation problem leading to ill-posed algebra system corrupting the solutions. This problem is noticeable especially for problems dealing with moving boundaries and discontinuous domains. It is obvious that free surface and multi-phase problems which are two different types of moving boundary problems, do not simulate more delicately by using the mesh-based methods. Meshless methods are new numerical approaches for solving partial differential equations. A regular or irregular distribution of nodes is used in meshless methods for the discretization of the domain, instead of traditional mesh applied in mesh-based methods. The most advantages of meshless methods in compare to mesh- based are: high flexibility and excellent accuracy especially for complex problems such as free surface flows and multi- phase problems. Furthermore, the cost of discretization is substantially reduced compared to the mesh-based methods. Hence, the interest in meshless methods to solve PDEs has grown rapidly in the recent years. Finite Point (FP) (Oñate et al., 2001), Hp Clouds and Finite Clouds (Liszka et al., 1996) are some type of meshless method.

The solution process in the meshless methods includes three main steps: Discritization of the domain, approximation the unknown nodal parameters and Minimization the residuals. Meshless methods can be classified to two major approaches for the approximating the unknown nodal parameters. The first approach uses the basic functions (shape function). The second approach has applied the integral method (kernel function). Weight function is normally used in integral methods to approximate the unknown nodal solutions which this method is a non- consuming procedure. In contrast, the order of accuracies obtained by this method for estimated functions is low, with respect to the methods used basic function approach. Smoothed Particle Hydrodynamic (SPH) (Gingold and Monaghan, 1977) and Moving Particle Semi-implicit (MPS) (Koshizuka et al., 1998) are the most familiar meshless methods used integral method. The computational cost of constructing shape functions is high in basic function approach but this approach doesn't waste this cost because the order of accuracy for obtained solutions is high remarkably. The high order of accuracy-especially for intensive gradient problems- supplies this trait that can be used less than the number of nodes to arriving the specific accuracy. Thereby, the computational cost is reduced, from this point of view. In terms of nature of the problems, can find out the basic function approach is more suitable for the intensive gradient problems such as- phase flow. Element Free Galerkin (EFG) (Belytschko et al., 1994) and Meshless Local Petrov-Galerkin (MLPG) (Atluri and Zhu, 1998) are two types of meshless methods using the basic function approach.

Recently, the mixed formulation technique is used in Discrete Least Squares Meshless (DLSM) (Arzani and Afshar, 2006 & Shobeyri and Afshar, 2010) method so called Mixed Discrete Least Squares Meshless (MDLSM) method for solving the linear elasticity problems (Amani et al., 2012). This method is excellently shown its high efficiency in solving the elliptic PDEs (Faraji et al., 2014). The MDLSM method is based on minimizing the least squares functional calculated at nodes which introduced on the problem domain and its boundaries. The least square functional is defined as the weighted summation of the squared residuals of the differential equation and its boundary conditions. A Moving Least Squares (MLS) approximation, a type of basic function approach, is used in this method to construct the meshless shape functions. The results are improved one order of accuracy by using the MDLSM method compare to DLSM method. In the current study, the MDLSM method is developed to solve the one dimensional advection- diffusion partial differential equation. The efficiency of suggested numerical method is evaluated with a number of numerical examples which their exact analytical solutions are available. The high efficiency and accuracy of MDLSM method to solve the advection- diffusion equation is then show by comparing the numerical and exact solutions.

2. Moving Least Squares (MLS) Shape Function

Various methods are used to produce the approximate functions in meshless methods such us Maximum Entropy (Sukumar, 2004), Partition of Unity (PU) (Sukumar et al., 2004) and Moving Kriging (MK) (Gu, 2003). Moving Least Squares shape function (Lancaster and Salkauskas, 1981) is a common effective method to approximate the unknown nodal solutions in meshless method. In this method, for each specified node assigns an influence domain and the approximate function (φ) calculates as following

$$\varphi(\boldsymbol{X}) = \sum_{i=1}^{s} p_i\,(\boldsymbol{X}) c_i(\boldsymbol{X}) = \boldsymbol{P}^T(\boldsymbol{X})\boldsymbol{c}(\boldsymbol{X}) \tag{1}$$

Basic function, $\boldsymbol{P}$, is determined by Eq. (2)

$$\boldsymbol{P}^T = [1, x, x^2, \dots, x^n]_{1\times s} \tag{2}$$

Where $\boldsymbol{X}$ is the nodal coordinates and $\boldsymbol{C}$ defines the coefficient of basic function, n and s are the order of basic function and the numbers of function components, respectively. This paper applied second order of basic function (n=2, s=6). In Eq. (3) a weighted two norm of a residual function is defined that it must be minimized to calculate the shape functions.

$$J = \sum_{j=1}^{num_s} w_j\left(\boldsymbol{X} - \boldsymbol{X}_j\right)\left(\boldsymbol{P}^T\left(\boldsymbol{X}_j\right)c(\boldsymbol{X}) - \bar{\varphi}_j\right)^2 \tag{3}$$

Where, num_s is total number of nodes in each influence domain. $\bar{\varphi}_j$ and w_j demonstrate nodal parameters and weight function in on *j*-th node, respectively. In the current study, Cubic spline is deployed as weight function defined by Eq. (4).

$$w_j(d) = \begin{cases} \frac{2}{3} - 4d^2 + 4d^3 & d \leq \frac{1}{2} \\ \frac{4}{3} - 4d + 4d^2 - \frac{4}{3}d^3 & \frac{1}{2} \leq d < 1 \\ 0 & d > 1 \end{cases} \quad (4)$$

$d = ||\boldsymbol{X}\text{-}\boldsymbol{X}j|| \,/dwj$, where dwj is the radius of influence domain. The shape function computed by minimizing the Eq. (3) displays in Eq (5).

$$\varphi(\boldsymbol{X}) = \boldsymbol{P}^T(X)\boldsymbol{F}^{-1}(X)\boldsymbol{M}(X)\bar{\boldsymbol{\varphi}} \quad (5)$$

Where, φ¯is the vector of the nodal parameters, F and M is defined as follows (in Eq. (6) and (7)).

$$\boldsymbol{F}(\boldsymbol{X}) = \sum_{j=1}^{num_s} w_j\,(X - X_j\,)\boldsymbol{P}(\,X_j\,)\,\boldsymbol{P}^T(\,X_j\,) \quad (6)$$

$$\boldsymbol{M}(\boldsymbol{X}) = [w_1(X - X_1)\boldsymbol{P}(X_1), \dots, w_{num_s}(X - X_{num_s})\boldsymbol{P}(X_{num_s})] \quad (7)$$

Assembling the Eq. (5-7) led to Eq. (8).

$$\varphi(\boldsymbol{X}) = \boldsymbol{A}^T(\boldsymbol{X})\bar{\boldsymbol{\varphi}} \quad (8)$$

Where, $\boldsymbol{A}$ is the shape function which it determines from Eq. (9).

$$\boldsymbol{A}^T = \boldsymbol{P}^T(\boldsymbol{X})\boldsymbol{F}^{-1}(\boldsymbol{X})\boldsymbol{M}(\boldsymbol{X}) \quad (9)$$

The first order of partial derivative of shape function is shown in Eq. (10) and (11).

$$\frac{\partial \boldsymbol{A}}{\partial x} = \frac{\partial \boldsymbol{P}^T}{\partial x}\boldsymbol{F}^{-1}\boldsymbol{M} + \boldsymbol{P}^T\frac{\partial \boldsymbol{F}^{-1}}{\partial x}\boldsymbol{M} + \boldsymbol{P}^T\boldsymbol{F}^{-1}\frac{\partial \boldsymbol{M}}{\partial x} \quad (10)$$

$$\frac{\partial \boldsymbol{A}}{\partial y} = \frac{\partial \boldsymbol{P}^T}{\partial y}\boldsymbol{F}^{-1}\boldsymbol{M} + \boldsymbol{P}^T\frac{\partial \boldsymbol{F}^{-1}}{\partial y}\boldsymbol{M} + \boldsymbol{P}^T\boldsymbol{F}^{-1}\frac{\partial \boldsymbol{M}}{\partial y} \quad (11)$$

3. MDLSM Method for Solving One-Dimensional Advection-Diffusion Equation

In this section the MDLSM method is developed for solving the one dimensional advection-diffusion equation defined as follows:

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = D\frac{\partial^2 U}{\partial x^2} + S(x) \quad (12)$$

Where C and U are the pollution concentration and the velocity of flow, respectively. D is the diffusion coefficient and $S(x)$ is the source term. The Eq. (12) can be written as follows:

$$\frac{C^{n+1}-C^n}{\Delta t} = (\alpha)\left(-U\frac{\partial C}{\partial x} + D\frac{\partial^2 C}{\partial x^2}\right)^{n+1} + (1-\alpha)\left(-U\frac{\partial C}{\partial x} + D\frac{\partial^2 C}{\partial x^2}\right)^{n} \tag{13}$$

α is the relaxation parameter and *Δt* display the time step. Assume:

$$C_x = \frac{\partial C}{\partial x} \tag{14}$$

The matrix form of Eq. (13) and (14) can be rewritten as follows:

$$\boldsymbol{L}(\boldsymbol{C}_i) = \boldsymbol{A_i}\boldsymbol{C},_{x_i} + \boldsymbol{B_i}\boldsymbol{C_i} \tag{15}$$

Where $\boldsymbol{L}$ is a differential operator. The matrices of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ vector are defined as follows for the ***i***-th node.

$$\boldsymbol{A_i} = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha D\Delta t \end{bmatrix}, \boldsymbol{B_i} = \begin{bmatrix} 0 & -1 \\ 1 & \alpha\Delta t {U_i}^n \end{bmatrix} \tag{16}$$

$$\boldsymbol{C_i}^T = [{C_i}^{n+1} \quad {C_{x_i}}^{n+1}] \tag{17}$$

Assume the following boundary conditions

$$\begin{cases} {C_i}^{n+1} = \bar{C}_i \\ {C_{x_i}}^{n+1} = \overline{C_{x_i}} \end{cases} \tag{18}$$

$(\boldsymbol{R}_\Omega)_i$ and $(\boldsymbol{R}_\Gamma)_i$ which show respectively the residuals of PDE and boundary on ***i***- th node, are defined as follows

$$\begin{cases} (\boldsymbol{R}_\Omega)_i = \boldsymbol{L}(\boldsymbol{C}_i) - \boldsymbol{f}_i; \quad \boldsymbol{f}_i = \left[0, S(x) + (1-\alpha)\Delta t\left(-U\frac{\partial C}{\partial x} + D\frac{\partial^2 C}{\partial x^2}\right)^n + C^n\right]_i^T \\ (\boldsymbol{R}_\Gamma)_i = \boldsymbol{I}(\boldsymbol{C}_i) - \overline{\boldsymbol{C}}_i; \quad \boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{cases} \tag{19}$$

The total residuals functional can be written as:

$$Re = \sum_{i=1}^{n} (\boldsymbol{R}_\Omega^T \boldsymbol{R}_\Omega)_i + \beta(\boldsymbol{R}_\Gamma^T \boldsymbol{R}_\Gamma)_i \tag{20}$$

β is the penalty coefficient that must be large enough. Minimizing the *Re* defined by Eq.(20) led to the nodal solutions.

4. Numerical examples

In this section the performance of suggested MDLSM method for solving the advection- diffusion equation is evaluated by benchmark numerical examples. A constant value of β $=10^8$ is used as the penalty coefficient in the all benchmark examples. The relaxation parameter (α) is 0.5. The two order polynomial basic function ($\boldsymbol{P}^T = [1, x, x^2]$) is applied to construct the MLS shape functions for the one dimensional problems. In the following examples source term is considered to be zero.

4. 1. Uniform input point source

In this example the advection- diffusion equation (Eq. (12)) is solved whit the following initial and boundary conditions.

$$\begin{cases} C(0,t) = C_0; & On\ t \geq 0 \\ C(x,0) = 0; & On\ x > 0 \\ C(\infty,t) = 0; & On\ t \geq 0 \end{cases} \tag{21}$$

The analytical solution of above problem was carried out by Ogata and Banks (1961) and the final results is presented by Eq. (22) The constant values used in this example are $C_0 = 100$, $U = 1$ and D=0.2.

$$\frac{C}{C_0} = \frac{1}{2}\left\{erfc\left(\frac{x-Ut}{2\sqrt{Dt}}\right) + e^{\frac{ux}{D}}\, erfc\left(\frac{x+Ut}{2\sqrt{Dt}}\right)\right\} \tag{22}$$

Figs. 1-3 compare the MDLSM results with available exact solutions for varied nodal distributions. The error of results is shown in table 1. The error norm is calculated by Eq. (23)

$$error\ norm = \frac{(C_{analytical}-C_{numerical})^T (C_{analytical}-C_{numerical})}{C_{analytical}{}^T C_{analytical}} \tag{23}$$

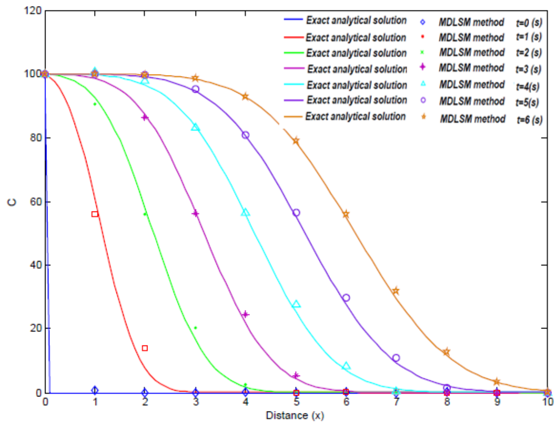

Fig. 1. First example using 11 nodes

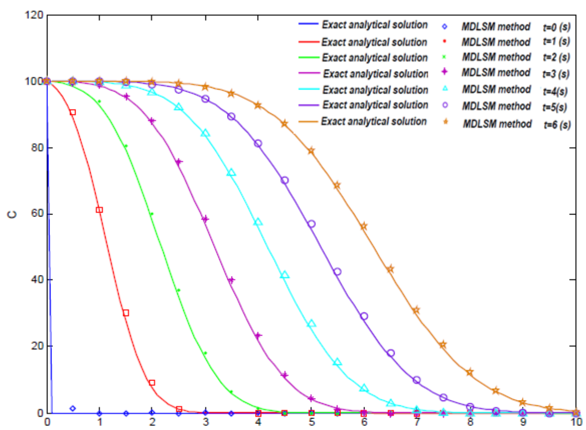

Fig. 2. First example using 21 nodes

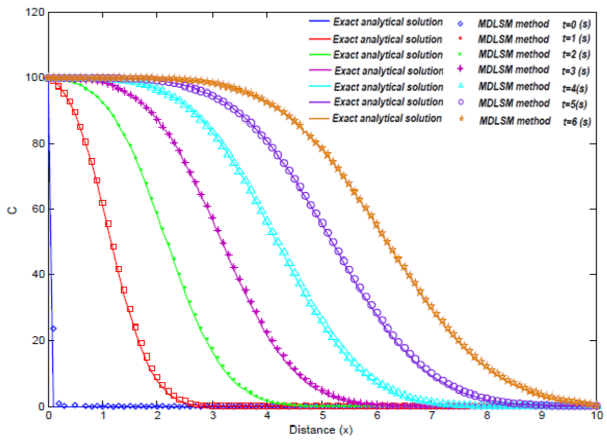

Fig. 3. First example using 101 nodes

Table. 1. The error of uniform input point source example

Number of nodes	11 nodes	21 nodes	101 nodes
Error norm	0.0347	0.0200	0.0144

4. 2. A constant pulse point source

In this example the advection- diffusion equation is solved on the following boundary and initial conditions.

$$\begin{cases} C(0,t) = \begin{cases} C_0\ ; & On \quad 0 \le t \le t_0 \\ 0\ ; & On \qquad t \ge t_0 \end{cases} \\ \quad C(x,0) = 0; \quad On\ x > 0 \\ \quad \frac{\partial C}{\partial x}(\infty, t) = 0; \quad On\ t \ge 0 \end{cases} \tag{24}$$

The analytical exact solution is presented as follows (Genuchten and Alves, 1982). The constant values used in this example are $C_0 = 20$, $t_0 = 1$, $U = 1$ and D=0.2.

$$\frac{C}{C_0} = \begin{cases} H(x,t) \quad On \ \ 0 \le t \le t_0 \\ H(x,t) - H(x, t - t_0)\ On \ \ t \ge t_0 \end{cases}$$

$$H(x,t) = \frac{1}{2}\left\{erfc\left(\frac{x - Ut}{2\sqrt{Dt}}\right) + e^{\frac{ux}{D}}\, erfc\left(\frac{x + Ut}{2\sqrt{Dt}}\right)\right\} \tag{25}$$

Figs. 4-6 compare the MDLSM results with available exact solutions for varied nodal distributions. The error of results is shown in table 2.

Table. 2. The error of a constant pulse point source

Number of nodes	11 nodes	21 nodes	101 nodes
Error norm	0.1126	0.0429	0.0159

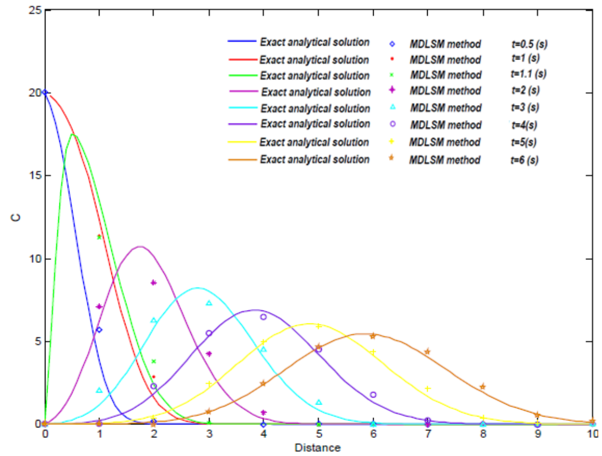

Fig. 4. Second example using 11 nodes

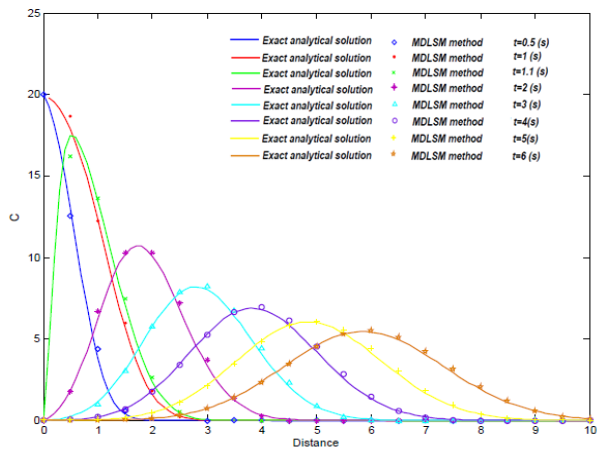

Fig. 5. Second example using 21 nodes

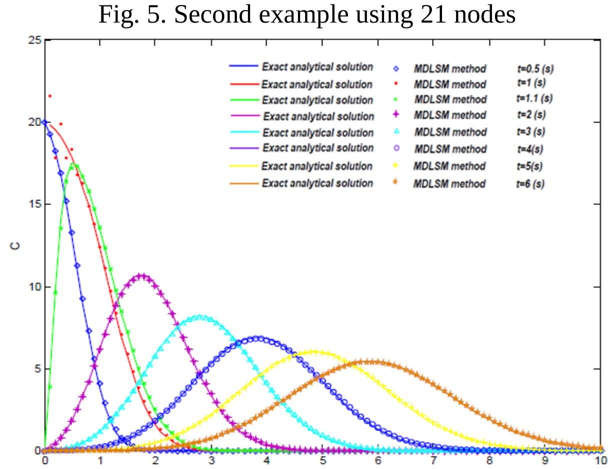

Fig. 6. Second example using 101 nodes

4. Conclusion

The mixed formulation technique was used in Discrete Least Squares Meshless (DLSM) method to introduce the Mixed Discrete Least Squares Meshless (MDLSM) method for solving the elliptic partial differential equations. The accuracy of numerical methods is improved one order by replacing the mixed formulation instead of the irreducible formulation. Application of this method was shown its high efficiency in solving the elliptic PDEs. The MDLSM method is based on minimizing the least squares functional calculated at nodes in the problem domain and its boundaries. The least square functional is defined as the weighted summation of the squared residuals of the differential equation and its boundary conditions. A Moving Least Squares (MLS) approximation, a type of basic function approach, is used in this method to construct the meshless shape functions. In the current study, the MDLSM method was developed to solve the one dimensional advection- diffusion equation as a parabolic PDE. The governed PDE for the diluted multi- phase flows such as the propagation of pollution can be illustrated by the advection- diffusion equation. The efficiency of suggested numerical method was evaluated by a number of standard numerical examples. Results obtained through proposed method were then compared with analytical solutions. The results were shown the high efficiency and accuracy for suggested MDLSM method to solve the parabolic PDEs.

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