Proceedings of the International Conference on Civil, Structural and Transportation Engineering
Ottawa, Ontario, Canada, May 4 – 5, 2015
Paper No. 301

Computation Time Assessment of a Galerkin Finite Volume Method (GFVM) for Solving Time Solid Mechanics Problems under Dynamic Loads

Saeed Reza SabbaghYazdi, Arwin Farhoud
KNToosi University of Technology, Department of Civil Engineering
Tehran, Iran
syazdi@kntu.ac.ir; e.farhoud@gmail.com

Abstract- In this paper, the Galerkin Finite Volume Method (GFVM) is developed for the investigation of time dependent problems in solid mechanics. The displacement field was computed using the GFVM by solving the two-dimensional force equilibrium equation on unstructured triangle meshes. Important features of this method are that it does not require matrix operations or time consumption. In order to show the efficiency of the suggested method, some time dependent mechanical problems are considered for the engineering structures such as a beam, which is subjected to dynamical loads. The computed transient deflections are used for evaluating the accuracy of the GFVM over Finite Element Method (FEM) and Meshless Petrov Galerkin Method (MLPG) Solvers. Comparison of the CPU time consumption of GFVM and FEM solvers shows that using the proposed method greatly reduces time consumption without affecting the accuracy of the results.

***Keywords*:** Galerkin Finite Volume Method, Solid Mechanic problems, Dynamics loads & response, CPU Time Consumption

1. Introduction

Time dependent problems are of considerable importance in different engineering and science fields. In a few dynamical problems, exact and analytical solutions can be found where the solutions are important for analyzing the system behaviour. Therefore, numerical methods are widely used in order to obtain the solutions and the responses of the systems. The traditional finite element method (FEM) was recognized as a dominating numerical method for elasto-static and elasto-dynamic problems, especially in practical engineering applications. Due to the problems encountered in the FEM, including volumetric locking during analysis of excessive displacements (Bijelonja et al., 2006) and high time consumption for time dependent problems, application of these alternative methods is on the increase. Also, with the advent of new numerical methods such as meshless methods (Nayroles et al., 1973, Wang et al., 2002), have provided an appropriate answer to the issues of the time dependent problems. The flaw these methods can be high computational cost, especially in problems with complex geometry and time dependent.

In the past, the finite volume method (FVM) was widely employed for solving Computational Fluid Dynamic (CFD) problems (Vestige et al., 2007). However, this trend has changed in recent years, so that FVM can now be applied for solving Computational Solid Mechanics (CSM) problems as well. In principle, because of the local conservation properties involved, FVM is in a strong position to effectively solve complicated geometries (Wheel et al., 1996, Slone et al., 2003).

Recently, the explicit Galerkin Finite Volume method has been used to compute stress-strain field as a result of the on structured triangular meshes (Sabbagh-Yazdi et al., 2012). GFVM is one of FVMs which has been introduced based on the procedure of defining subdomains and integrating governing equations.
In the present study, GFVM is proposed for solving the time dependent problems in solid mechanics. This method is utilized for computing stress-strain fields on the same unstructured linear triangular elements.

Before presenting the dynamic numerical results, the static study with further computational results is presented. In order to show the accuracy and the efficiency of the proposed technique, the results are compared with the available results in the literature. Therefore, the suggested method can be used as a base method for analyzing the time dependent problems with more simplicity and much less cost than other numerical methods in the solution.

2. The Basic Equations of Elasto-Dynamics

The initial boundary value equations of elasto-dynamics for small displacements are as follows:

$$\rho\frac{\partial^2 u_i}{\partial t^2}=\frac{\partial \sigma_{i1}}{\partial x_i}+b_i \qquad i=1,2 \tag{1}$$

Where, ρ is the material density, u_i is the displacement in i direction, σ is the stress tensor, t is the time and b is the body force. For 2D problems and in an x-y coordinate system:

$$\sigma_{11}=\left(c_{11}\frac{\partial u_1}{\partial x_1}+c_{12}\frac{\partial u_2}{\partial x_2}\right) \tag{2}$$

$$\sigma_{21}=c_{33}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) \tag{3}$$

$$\sigma_{21}=c_{33}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) \tag{4}$$

$$\sigma_{22}=\left(c_{21}\frac{\partial u_1}{\partial x_1}+c_{22}\frac{\partial u_2}{\partial x_2}\right) \tag{5}$$

Where c is the stiffness matrix. For the plane stress problem, constants are:

$$c=\frac{E}{(1-\vartheta^2)}\begin{bmatrix}1 & \vartheta & 0\\ \vartheta & 1 & 0\\ 0 & 0 & \frac{1-\vartheta}{2}\end{bmatrix} \tag{6}$$

Here, ϑ is the Poisson's ratio and E is Young's modulus of elasticity.

2. 1. Galerkin Finite Volume Formulation

Multiplying the residual of the Cauchy equation by the test function, ω and integrating over a sub_domain, Ω (see Figure 1), the terms containing spatial derivatives can be integrated by part. The governing equation may be written as:

$$\int_\Omega \omega.\rho\frac{\partial^2 u_i}{\partial t^2}d\Omega=[\omega.\vec{F_i}]_\gamma-\int_\Omega(\vec{F_i}.\vec{\nabla}\omega)\,d\Omega+\int_\Omega \omega.b_i d\Omega \tag{7}$$

Where the stress vector in i direction is defined as $\vec{F_i}=\sigma_{i1}\hat{\imath}+\sigma_{i2}\hat{\jmath}$.

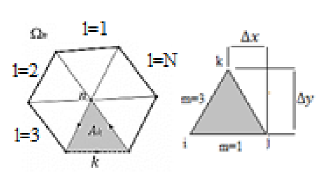


Fig. 1. Triangular element with area A_k within the sub-domain Ω_n

According to the Galerkin method, the weighting function, ω can be chosen to be equal to the interpolation function ϕ. In finite element method, this function is systematically computed for the desired element type and is called the shape function. For a triangular type element (with three nodes), the linear shape functions ϕ_k take the value of unity at a desired node, n and zero at other neighbouring nodes, and k of each triangular element (see Figure 1). Therefore, the summation of the term $[\omega.\vec{F_i}]_\gamma$ over the boundary of the subdomain Ω_n is zero. So, the right- hand side of equation (4) can be discretized as:

$$\int_\Omega \left(\vec{F}_i.\vec{\nabla}\emptyset\right)d\Omega = \iint(\sigma_{i1}\frac{\partial\emptyset}{\partial x} + \sigma_{i2}\frac{\partial\emptyset}{\partial x})dxdy = -\frac{1}{2}\sum_{k=1}^{N}(\vec{F}_i.\vec{\Delta}l)_k \tag{8}$$

Where $\vec{\Delta}l_k$ is the normal vector of the side k opposite to the node, n and $\vec{F}_i$ is the i direction piecewise constant stress vector at the center of the element associated with the boundary side k. For cases with a body force term, the second integral in the right-hand side of equation (4) can be discretized as:

$$\int_\Omega \emptyset b_i d\Omega \approx \frac{\Omega_n}{3} b_i \tag{9}$$

The left-hand side of equation (4) can be written in discretized form as:

$$\frac{\delta^2}{\delta t^2}(\int_\Omega \emptyset u_i d\Omega) \approx \frac{\Omega_n}{3}\frac{d^2 u_i}{dt^2} \tag{10}$$

By applying the finite difference method concept in the procedure of discretization of the left-hand side's transient term, the time derivative of i direction displacement u_i in equation (4) can be discretized as:

$$\rho\frac{\Omega_n}{3}\frac{\partial^2 u_i}{\partial t^2} = \rho(\frac{u_i^{t+\Delta t} - 2u_i^t + u_i^{t-\Delta t}}{(\Delta t)^2})\frac{\Omega_n}{3} \tag{11}$$

Eventually, using equations (5), (6), (7) and (8), the discrete form of the Cauchy's equilibrium equations can be formulated as:

$$(\frac{u_i^{t+\Delta t} - 2u_i^t + u_i^{t-\Delta t}}{(\Delta t)^2})_n = \frac{3}{2\rho\Omega_n}\sum_{k=1}^{N}(\sigma_{i1}\Delta y - \sigma_{i2}\Delta x)_k + \frac{3}{\rho\Omega_n}(b_i\frac{\Omega_n}{3}) \tag{12}$$

Where $(u_i^{t+\Delta t})_n$ is the displacement of node n at $t+\Delta t$ computational stage in i direction. Considering direction i=1 as x and i=2 as y, the stress$\tilde{\sigma}_{i1}$, $\tilde{\sigma}_{i2}$ as:

$$\tilde{\sigma}_{xx} = \frac{1}{A_k}\sum_{m=1}^{N}(c_{11}u_x\Delta y - c_{12}u_y\Delta x)_m \tag{13}$$

$$\tilde{\sigma}_{xy} = \frac{1}{A_k}\sum_{m=1}^{N}(c_{33}u_x\Delta y - c_{33}u_y\Delta x)_m \tag{14}$$

$$\tilde{\sigma}_{yy} = \frac{1}{A_k}\sum_{m=1}^{N}(c_{21}u_x\Delta y - c_{22}u_y\Delta x)_m \tag{15}$$

Where c is the stiffness matrix and A_k is the area of the triangular element (with m=3 sides) associated with the boundary side k of the sub-domain Ω_n (see Figure 1).

2. 2. Time Integration

In order to have stable explicit solution, the Courant's number must be less than 1. According to proposed relation from reference (Sabbagh-Yazdi et al,. 2011), the time step has been limited to following amount:

$$\Delta t_n < \frac{r_n}{s_t} \tag{16}$$

The parameter r_n is ratio of area to perimeter in each control volume.

$$r_n = \frac{\Omega_n}{P_n} \quad , \quad P_n = \sum_{k=1}^{Nedge} (L)_k \tag{17}$$

Where Ω_n and P_n is area and perimeter of the control volume, respectively. S_t is the speed of information transition which is calculated from equation (18):

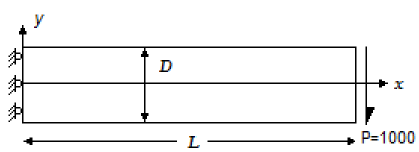
$$S_t = \sqrt{\frac{E}{\rho(1-\vartheta^2)}} \tag{18}$$

3. Numerical Results

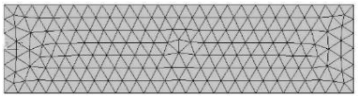
In this section, numerical examples for the forced vibration of the 2-D structures and the dynamic response of the structures are studied to examine the presently developed GFVM solution in the dynamic analysis and demonstrate its efficiency. However, before analyzing the dynamic response, the GFVM solver is verified by comparison of computed results with analytical solution of a cantilever Timoshenko beam that carries an end load P (Timoshenko et al., 1970). Then convergence and the CPU time of the GFVM for time dependent problem are compared with other numerical methods. All the computations are performed using a computer with Intel Core(TM) Duo T2450 2GHz CPU, with 1 GB RAM memory.

3. 1. Verification

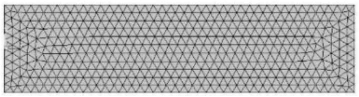
The performance of GFVM for the static stress analysis has been studied by Lam et al. (2004) in detail. In this example, consider a 2D cantilever beam with unit thickness as shown in Fig. 2. The beam is considered in the plane stress state. The material properties and the dimensions of the beam are assumed to be:$E = 3 \times 10^7$,$\upsilon = 0.3$, $\rho = 1.0$, $L = 48$ and $D = 12$ that carries an end load P and all units are in SI.



(a)



b) With 208 nodes and 352 elements



c) With 492 nodes and 882 elements

Fig. 2. Cantilever beam for static analysis: (a) Geometry of beam (b) Coarse mesh (c) Fine mesh

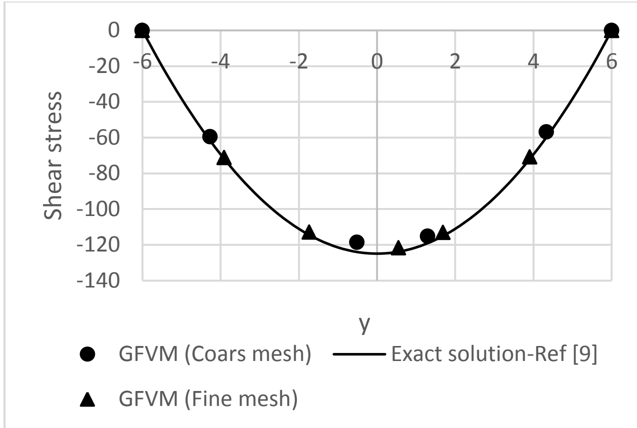


Fig. 3. Shear stress at the section $x = {}^{L}/_{2}$ of the cantilever beam

Some results of the static analysis of a 2-D structure as shown in Fig. 3 are simply presented here. The behaviours of the cantilever beam shown in Fig.3 have been studied using conventional adaptive GFVM modelling. Variation of the shear stress at the section $x = \frac{L}{2}$ of the cantilever beam obtained by the GFVM is compared with the analytical solution and yields good agreement.

3. 2. Evaluation of a Timoshenko Beam Subjected to Dynamic Loadings

In this section, the cantilever beam in Section 3-1 is subjected to a parabolic traction at the free edge by the function $P = 1000g(t)$where $g(t)$ is the function of time. Three kinds of the dynamic loading are considered, namely transient loading, step loading and simple harmonic loading (see Fig. 4 (a)-(c)). Other parameters of this beam are the same as that given in the Section 3-1. In this dynamic analysis, the time step 1×10^{-4} and 1×10^{-5} are used and the damping is neglected.

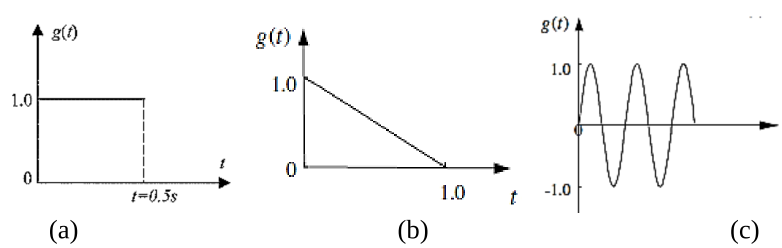


Fig. 4. Function of time: (a) step loading, (b) triangular pulse and (c) harmonic loading

3. 2. 1. Step Function

In Fig. 5, a comparison between the transient responses for the middle points of the free edge of the beam obtained by commercial software based on FEM, present algorithm and MLPG solution by Gu et al. (2004) are presented. As the figure shows, GFVM solution is a good agreement between the results. Also, The CPU time is plotted in table. 1. From this table, it is observed that FEM and MLPG are also computationally expensive and GFVM needs a much less CPU-time than the FEM and MLPG.

Table. 1. Comparison of CPU-Time in numerical methods for step loading

	Number of Node	Number of Element	Δt (s)	CPU-Time (s)
GFVM	208	352	0.0001	4.94
GFVM	208	352	0.00001	47.24

Commercial software (FEM)	208	352	0.0001	989.69

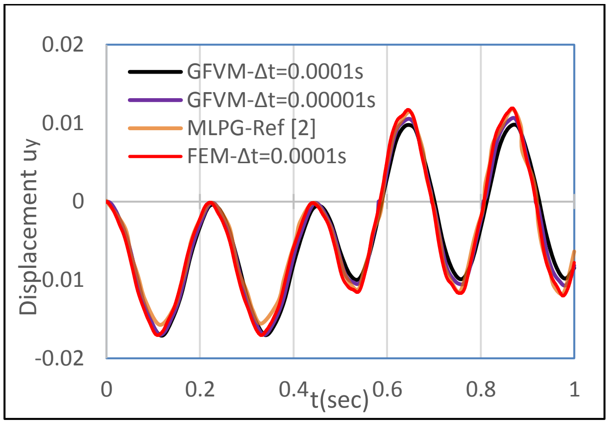


Fig. 5. Transient displacements u_y for step loading

3. 2. 2. Triangular Pulse Function

In this case, the function g(t) is shown in Fig. 5(b). Comparison of numerical results for the displacement and CPU time in this case are shown in Fig. 6 and table. 2, respectively.

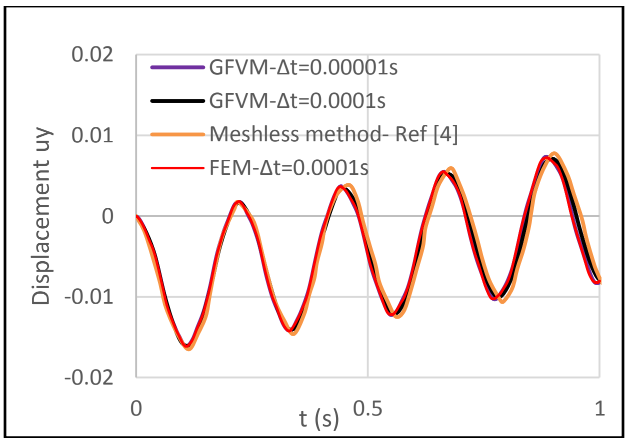


Fig. 6. Transient displacements u_y for triangular pulse loading

Table. 2. Comparison of CPU time in numerical methods for triangular pulse loading

	Number of Node	Number of Element	Δt (s)	CPU-Time (s)
GFVM	208	352	0.0001	6.71
GFVM	208	352	0.00001	68.39
Commercial software (FEM)	208	352	0.0001	952.90

3. 2. 3. Simple harmonic loading

Let the function$g(t) = \sin(\omega_f t)$, where ω_f is the frequency of dynamic loading. For this example, the value of $\omega_f = 27$ is used. The obtained results for the displacements and CPU-time are plotted in Fig. 7 and table. 3, respectively.

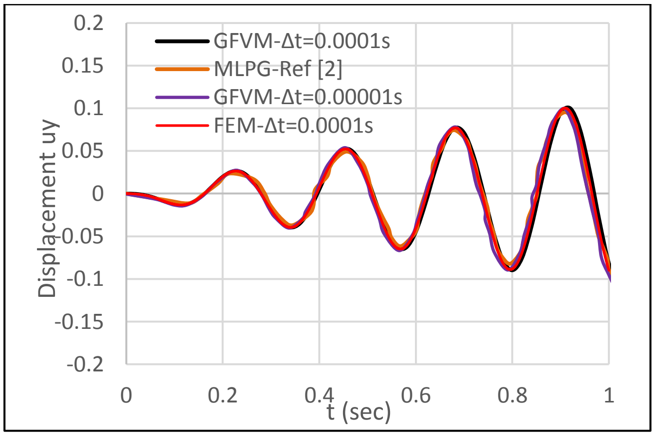


Fig. 7. Transient displacements u_y for harmonic loading

Table. 3. Comparison of CPU Time in numerical methods for harmonic loading

	Number of Node	Number of Element	$\Delta t\ (s)$	CPU-Time (s)
GFVM	208	352	0.0001	6.9
GFVM	208	352	0.00001	47.24
Commercial software (FEM)	208	352	0.0001	962.75

4. Conclusion

In this paper, the Galerkin Finite Volume Method (GFVM) is developed for the investigation of time dependent problems in solid mechanics. Several numerical examples are analyzed by the suggested method and the obtained results were compared with the computed results of commercial software based on FEM solid mechanics solver and Meshless Petrov Galerkin Method (MLPG). From accuracy point of view, the comparison of the results shows that the GFVM is a comparative method with FEM and MLPG for a number of solid mechanic bench tests under dynamic loading. However, since the GFVM does not require matrix operations, its CPU time consumption for computation is considerably less than FEM solver (which is faster than MLPG). This feature makes GFVM a suitable solver for analysing structural cases under long term dynamic loads.

References

Bijelonja I., Demirdžić I., Muzaferija S. (2006). A finite volume method for incompressible linear elasticity. Computer Methods in Applied Mechanics and Engineering, 195, pp. 6378-6390.

Gu Y., Liu G. (2001). A meshless local Petrov–Galerkin method for free and force vibration analysis for solids. Computational Mechanic, 27, pp. 188–98.

Lam K.Y., Wang Q.X., Li H. (2004). A novel meshless approach—Local Kriging (LoKriging) method with two-dimensional structural analysis. Comput. Mech, 33 (3), pp. 235–244.

Liu G., Gu G. (2005). An introduction to meshfree methods and their programming. Netherland, Springer.

Nayroles B., Touzot G., Villo P. (1973). Generalization the finite element method: diffuse approximation and diffuse elements. Computational Mechanics. 10, pp. 307–318.

Slone, Bailey C., Cross M. (2003). Dynamic solid mechanics using finite volume methods. Applied Mathematical Modelling, 27, pp. 69-87.

Sabbagh-Yazdi S.R., AliMohammadi S., Pipelzadeh M.K. (2012). Unstructured finite volume method for matrix free explicit solution of stress-strain field in two-dimensional problems with curved boundaries in equilibrium condition. Applied Mathematical Modeling, 36(5), pp. 2224-2236.

Sabbagh-Yazdi S.R., AliMohammadi S. (2011). Performance evaluation of iterative GFVM on coarse unstructured triangular meshes and comparison with matrix manipulation based solution methods, Journal of Scientia Iranica, 18(2), pp. 131-138.

Timoshenko S.P., Goodier J.N. (1970). Theory of Elasticity. McGraw-Hill, New York.

Vestige H.K., Malalasekera W. (2007). An Introduction to Computational Fluid Dynamics: The Finite Volume Method, Prentice Hall.

Wang J., Liew K.M., Tan M.J. (2002). Rajendran S. Analysis of rectangular laminated composite plates via FSDT meshless method. International Journal of Mechanical Science. 44, pp. 1275–1293.

Wheel M.A., (1996). A geometrically versatile finite volume formulation for plane elastostatic stress analysis. Journal of Strain Analysis, 31(2), pp. 111-116.