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Linear Behaviour of Thin Walled Open Sections using GBT

Rakhi Jain, S. Arul Jayachandran
Indian Institute of Technology, Madras
Chennai, India
jainrakhie7@gmail.com; arulsteel@gmail.com

Abstract- Finite strip (FSM) and finite element method (FEM) techniques are typically used for cold formed steel analysis. Software give quite satisfactory results in analysis of cold formed steel structures but they lack physical interpretation and understanding of the mechanics involved. To get a physical intuition of the internal mechanics involved, GBT was introduced by Schardt in 1966. Generalized Beam Theory (GBT) has this distinguishing feature of discretizing the member deformed configuration into contributions of cross-section deformation modes, which can be global, local-plate or distortional. In this work, a framework has been used to analyze cross sectional behaviour of any generic open section using first order GBT. Elementary mode shapes and stiffness matrices are developed. As elementary mode shapes are coupled in nature, simultaneous diagonalization and normalization process is performed to uncouple elementary modes and then GBT mode shapes and stiffness matrices are established. Even after this process, some matrices are left non-diagonalized. In order to observe the effect of coupling terms in the GBT matrices which are influencing the results, the GBT fundamental equation is solved for both coupled and uncoupled nature. Also, to see the impact of each mode, analysis is run for different lengths and finally modal participation is obtained.

***Keywords*:** Cold Formed Steel, Generalized Beam Theory, Linear Static Analysis

1. Introduction

The trivial way of analyzing steel structures is by using Finite element analysis, until, Schardt[1] developed “Generalized beam theory” (GBT) in 1966, by which one can accurately and intuitively analyze any cold formed steel structure in comparatively lesser time. With increasing complexities in the design, accurate understanding of the deformation pattern is of utmost priority and that too in less time.

Full length simulations take much longer to fully analyze any section.

The basic beam equation describes four types of deformation of any member viz axial, bending about major axis, bending about minor axis, and torsion. These are called rigid body modes as there is no change in outline of the cross section. But due to high slenderness ratio of the plate elements, there is a tendency to fail much below the yield point due to the local and distortional effects which causes distortional buckling, flexural buckling, torsion buckling and their combinations along with global buckling of the member. These elements also contribute to post buckling strength of the section depending upon the mode of buckling. Hence, the other deformation modes play major role where cross section deformation occurs. GBT is one such methodology which can be used to study all the deformation modes which are important in analysis and can be considered in design.

A three phase design has been developed that takes cross sectional data as input and gives deformation pattern as the final output.

Initially, cross sectional analysis is done, in which modes shapes are obtained through diagonalization process. Next, we solve the GBT equation using finite element formulation. Finally, deformation pattern is established by calculating parameters like deflection, modal participation etc. Each phase is broadly discussed in the following sections. Although the framework can be used to analyze any open section, for the purpose of clear understanding, we perform the full procedure on a lipped channel section with web stiffener in this paper.

2. Methodology

The basic equation of GBT [2,3,4] is

$$EC_k\phi''''_k - GD_k\phi''_k + B_k\phi_k = q_k \quad (1)$$

C_k, D_k, B_k= section properties of mode k
ϕ_k= generalized deformation in mode k
q_k= distributed load applicable to mode k

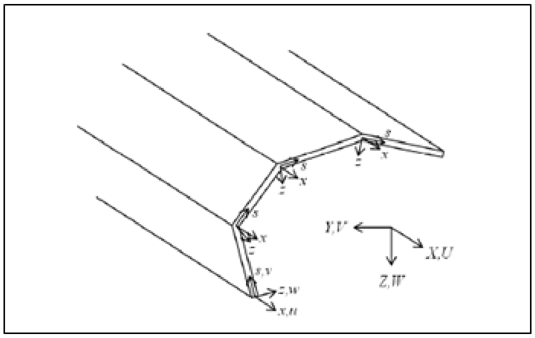

Fig. 1. Displacement components of a cross section [6]

$$u(x,s) = \sum_{1}^{m} u_k(s)\emptyset_{k,x}(x) \quad (2)$$

$$v(x,s) = \sum_{1}^{m} v_k(s)\emptyset_{k,x}(x) \quad (3)$$

$$w(x,s) = \sum_{1}^{m} w_k(s)\emptyset_{k,x}(x) \quad (4)$$

2. 1. Cross Section Analysis

The first step in GBT begins with cross section analysis which results in finding modes of deformation called GBT deformation modes and cross section properties related to each mode.

The basic displacements in a deformed cross-section mode consists of the following (fig.1):

a) Warping component (out of plane deformation)–u^e
b) Flexural component (in-plane deformation) –w^e
c) Transverse component (deformation parallel to outline of section) –v^e

There are two types of elementary modes.

a) Warping elementary modes
b) Flexural elementary modes

The mentioned analysis requires discretization of nodes as:

a) End nodes
b) Intermediate nodes
c) Natural nodes

2. 1. 1. Warping Elementary Modes

All end and natural nodes which are independent in nature contribute to a warping mode. Each of these nodes is given a displacement of one unit with zero displacement at other warping nodes [3]. This deformation varies linearly along the element length. Hence, using interpolation method, displacement at every point of the element under consideration can be calculated.

$$u^e = u_0^e + u_1^e . s \quad ; \quad v_r^e = \frac{1}{b_r}[u_r^e - u_{r+1}^e] \tag{5}$$

$$w_{r-1}^e = \frac{v_r^e}{sin\Delta\alpha_r} - \frac{v_{r-1}^e}{tan\Delta\alpha_r} \quad ; \quad w_r^e = \frac{v_r^e}{tan\Delta\alpha_r} - \frac{v_{r-1}^e}{sin\Delta\alpha_r} \tag{6}$$

2. 1. 2. Flexural Elementary Modes

All dependent and independent end nodes along with intermediate nodes are the ones which produces flexural elementary modes. At each of the concerned node, in-plane unit displacement is given with zero displacement at other nodal points. The displacements along each element can be found out using folded plate theory and are cubic in nature. The flexure equation is written as:

$$w^e = w_0^e + w_1^e . s + w_2^e . s^2 + w_3^e . s^3 \text{ [5]} \tag{7}$$

Where s varies along the length of each element.

$$\mathrm{C_{ik}^e} = \mathrm{Eh} \int u_i^e \, u_k^e ds + \frac{Eh^3}{12(1-v^2)} \int w_i^e w_k^e \, ds \text{ [5]} \tag{8}$$

$$\mathrm{D_{ik}^e} = \frac{Gh^3}{3} \int w_i^e w_k^e \, ds - \frac{vEh^3}{12(1-v^2)} \int (w_i^e w_{k,ss}^e + w_k^e w_{i,ss}^e) \, ds \text{ [5]} \tag{9}$$

$$\mathrm{B_{ik}^e} = \frac{Eh^3}{12(1-v^2)} \int w_{i,ss}^e w_{k,ss}^e \, ds \text{ [5]} \tag{10}$$

The above equations are used to obtain elementary stiffness matrices from elementary flexure and warping modal displacements.

2. 1. 3. GBT Mode Shapes

The B^e, C^e, D^e elementary matrices representing section properties are coupled with each other. Hence, in order to make them uncoupled, diagonalization and normalization is done which results into ‘A’ matrix i.e. transformation matrix along with uncoupled B C D matrices.

1. Initially, B and C are diagonalized with respect to each other. The eigenvalue problem results in nd eigenvalues and nd×nd right eigenvectors termed as a_k.

$$([B^e] - \lambda_k E[C^e])\{a_k\} = \{0\} \tag{11}$$

A^1 is picked for the corresponding right eigenvectors for which eigenvalues are zero. These need to be further digonalized.

$$[B^{e1}] = [A^1]^T[B^e][A^1] \quad ; \quad [C^{e1}] = [A^1]^T[C^e][A^1] \quad ; \quad [D^{e1}] = [A^1]^T[D^e][A^1] \tag{12}$$

2. Secondly, diagonalization is performed on D^{e1} with respect to C^{e1} matrix which results in three null eigenvalues corresponding to axial and two bending modes and one positive eigenvalue corresponding to the torsion mode.

$$(G[D^{e1}] - \lambda_k E[C^{e1}])\{b_k\} = \{0\} \tag{13}$$

A^2 is the picked matrix corresponding to the three null eigenvalues, which is further processed.

$$[D^{e2}] = [A^2]^T[D^{e2}][A^2] \; ; [C^{e2}] = [A^2]^T[C^{e2}][A^2] \tag{14}$$

3. Finally, the two bending modes are separated by solving the below eigenvalue problem. This leads to one null eigenvalue corresponding to the axial extension mode, while the two remaining eigenvalue corresponds to the major and minor axis bending modes.

$$([X] - \lambda_k E[C^{e2}])\{c_k\} = \{0\} \tag{15}$$

'A' matrix is obtained by concatenating a_k, b_k, c_k matrices which needs to be normalized to make sure that the matrix corresponds to the unit displacements such that, for the first mode, the axial displacement is unity and for bending modes, the displacements are unity in global transverse directions [1].

Table 1. Normalization Conditions

Axial Extension	$u_1(s) = u_1 = 1$
Bending Major Axis	$\sqrt{v_2(s)^2 + w_2(s)^2} = 1$
Bending Minor Axis	$\sqrt{v_2(s)^2 + w_2(s)^2} = 1$
Torsion	$w_{4,s} = \pm 1$
For local modes and distortional modes	$max\left\{\sqrt{v_i(s)^2 + w_i(s)^2}\right\} = 1$

The 'A' matrix is used for transforming the elementary stiffness matrices to GBT stiffness matrices.

$$[B] = [A]^T[B^e][A]; \qquad [D] = [A]^T[D^e][A]; \qquad [C] = [A]^T[C^e][A] \tag{16}$$

$$u^g(s) = A.u^e(s); \qquad v^g(s) = A.v^e(s); \qquad w^g(s) = A.w^e(s) \tag{17}$$

Where, $u^g(s), v^g(s), w^g(s)$ are displacement components of GBT mode shapes.

2. 2. Member Analysis

GBT modes as obtained from the cross sectional analysis gives a visualization of possibilities of number of deformed shapes of a cross section occurring along the member with a structural meaning. But actual participation of each mode can be established through member analysis. At this stage, loading conditions and boundary conditions are considered.

2. 2. 1. Calculation of Modal Loads

Surface loads and body forces are required to be converted into line loads to incorporate in the GBT procedure. Loads are calculated on the application area and are distributed equally among the nodes which are then converted into global loads. The nodal loads obtained are then converted to modal loads.

$$q_k = \sum_r^{n+1}(q_{Y.r}V_{r.k} + q_{Z.r}W_{r.k}) \quad ; \quad q_{x.k,x} = \sum_r^{n+1}(q_{X.r,X}U_{r.k}) \tag{18}$$

The above equations are for modal loads in transverse and axial directions.

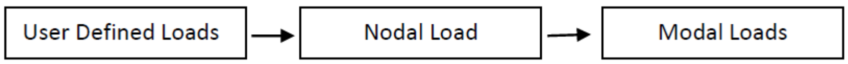

Fig. 2. Process of Obtaining Modal Loads

2. 2. 2. Finite Element Formulation for Linear Static Analysis

Using finite element method, numerical solutions are obtained in two ways

a. Uncoupled solution
b. Coupled solution

In case of uncoupled solution, GBT equation is separately solved for each mode after arranging elementary stiffness matrix of all beam elements along the member length into global stiffness matrix. The individual stiffness matrix so obtained for each mode is solved for the amplitude function. The diagonal elements of GBT stiffness matrix are only considered in this solution. Since D Matrix contains some coupled elements, solving in coupled manner will give a better solution.

Coupled solution is obtained by generating stiffness matrix considering all modes for a single beam element and thereby obtaining final stiffness matrix. Since both diagonal and non-diagonal terms are considered, higher accuracy in results can be seen. Elementary stiffness matrix for a beam element is:

$$K = K_C^e + K_D^e + K_B^e \tag{19}$$

$$= E * C_{ij}\begin{bmatrix} \frac{12}{L_e^3} & \frac{6}{L_e^2} & -\frac{12}{L_e^3} & \frac{6}{L_e^2} \\ \frac{6}{L_e^2} & \frac{4}{L_e} & -\frac{6}{L_e^2} & \frac{2}{L_e} \\ -\frac{12}{L_e^3} & -\frac{6}{L_e^2} & \frac{12}{L_e^3} & -\frac{6}{L_e^2} \\ -\frac{6}{L_e^2} & \frac{2}{L_e} & -\frac{6}{L_e^2} & \frac{4}{L_e} \end{bmatrix} + G * D_{ij}\begin{bmatrix} \frac{6}{5L_e} & \frac{1}{10} & -\frac{6}{5L_e} & \frac{1}{10} \\ \frac{1}{10} & \frac{2L_e}{15} & -\frac{1}{10} & -\frac{L_e}{30} \\ -\frac{6}{5L_e} & -\frac{1}{10} & \frac{6}{5L_e} & -\frac{1}{10} \\ \frac{1}{10} & -\frac{L_e}{30} & -\frac{1}{10} & \frac{2L_e}{15} \end{bmatrix} + B_{ij}\begin{bmatrix} \frac{13L_e}{35} & \frac{11L_e^2}{210} & \frac{9L_e}{70} & -\frac{13L_e^2}{420} \\ \frac{11L_e^2}{210} & \frac{L_e^3}{105} & \frac{13L_e^2}{420} & -\frac{L_e^3}{140} \\ \frac{9L_e}{70} & \frac{13L_e^2}{420} & \frac{13L_e}{35} & -\frac{11L_e^2}{210} \\ -\frac{13L_e^2}{420} & -\frac{L_e^3}{140} & -\frac{11L_e^2}{210} & \frac{L_e^3}{105} \end{bmatrix}$$

The above equation is the stiffness matrix for single mode for uncoupled solution method.

$$f = f_{q,k} + f_{qx,k} = q_k\begin{bmatrix} L_e/2 \\ L_e^2/12 \\ L_e/2 \\ -L_e^2/12 \end{bmatrix} + q_{x,k}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{20}$$

The three element stiffness matrix K_C^e, K_D^e, K_B^e are converted in global form for the whole beam member, hence solving GBT equation for each mode and finding out deformation from each mode separately using equation

$$[K_C + K_B + K_D]\{d\} = [f] \tag{21}$$

Since the fundamental GBT equation involves fourth ordered derivatives, four boundary conditions are needed, two at each end. Table 2 represents the boundary conditions for fixed, guided and pinned support.

Table. 2. Boundary Conditions

Fixed	Guided	Pinned
$\emptyset_k(0) = 0$	$\emptyset_{k,x}(0) = 0$	$\emptyset_k(0) = 0$
$\emptyset_{k,x}(0) = 0$	$W_{k,x}(0) = 0$	$W_k(0) = 0$

2. 3. Calculation and Assessment of Result

2. 3. 1. Modal Participation

$$P_i = \frac{w_i(s_P)\,\emptyset(x_P)}{\sum_{k=1}^{m} w_k(s_P)\emptyset(x_P)} \times 100\% \qquad (22)$$

The above equation expresses the participation of i^{th} mode in the vertical deformation of the member over summation of contribution of all other modes. It gives an intuition of different modes occurring for various lengths. Since the analysis distinguishes the major mode shapes for a particular system, hence gives us the flexibility to choose the relevant modes and exclude the irrelevant ones.

3. Linear Static Analysis on Example Section

In order to demonstrate GBT application, linear static analysis of a lipped channel section with web stiffener a shown in fig. 3 with line load of 5KN/m on the fourth node point of top flange of the section is performed. Channel with stiffener in web increases capacity of the unstiffened sections which are used in floor constructions, hallway constructions, flooring for modular buildings, roof arches, etc.

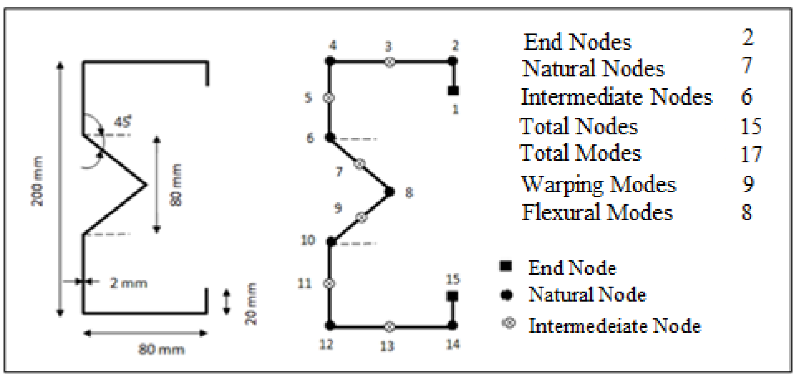

Fig. 3. Details of Example Cross Section

3. 1. Cross-Section Analysis

In fig. 4, the first four mode shapes belongs to global mode shapes i.e. Axial, Bending in both directions and Torsion. The modes presented from 5 to 9 are distortional modes.

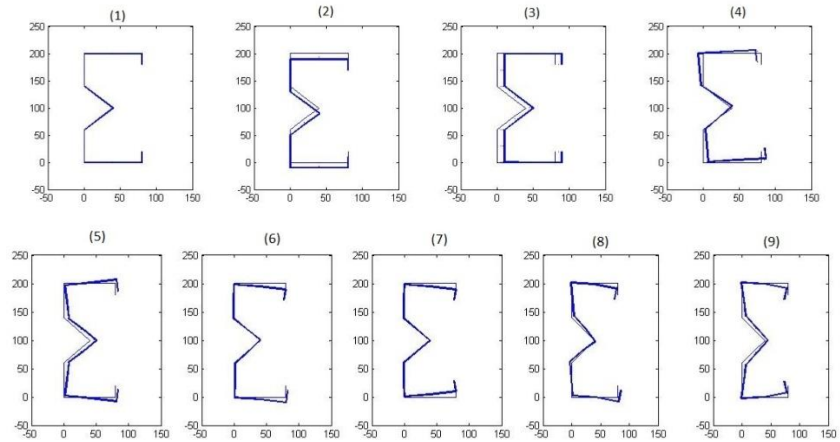

Fig. 4. GBT Mode Shapes

3. 2. Loading Criteria

As shown in fig. 5, the cantilever member under consideration is of 2.4m length. The applied load along with self-weight is distributed to all nodes as shown in table 3 and then converted to modal loads.

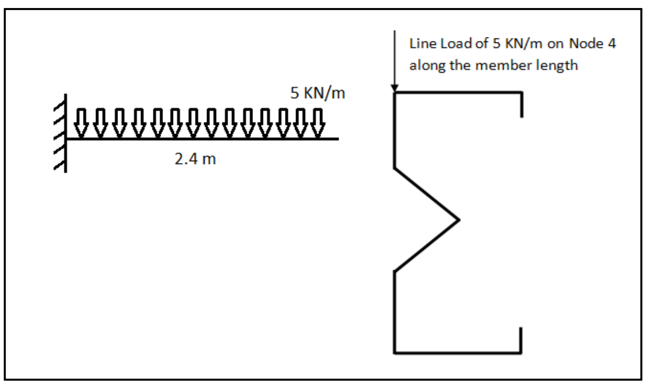

Fig. 5. Loading Diagram

Table. 3. Nodal Loads

Node Point	Line load (KN/m)	Self Weight (KN/m) 10^{-4}	Total load distribution (KN/m) 10^{-4}
1	0	-15.31	-15.31
2	0	-45.94	-45.94
3	0	-61.26	-61.26
4	-5	-53.60	-50053.60
5	0	-45.94	-45.94
6	0	-66.29	-66.29
7	0	-86.64	-86.64
8	0	-66.29	-66.29
9	0	-45.94	-45.94
10	0	-53.60	-53.6
11	0	-61.26	-61.26
12	0	-45.94	-45.94
13	0	-15.31	-15.31

3. 3. Comparison between Coupled and Uncoupled Solution

Clearly from the below results, we can see that there is no modal participation in case of uncoupled solution since the loading is in vertical direction. However, the coupled solution considers effect of each mode on the other and therefore some participation of the third mode can be seen in Table 4.

Table. 4: Comparison between Coupled and Uncoupled solution

	Uncoupled			**Coupled**		
Mode k	0.25L	0.5L	0.75L	0.25L	0.5L	0.75L
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	2.0232	6.7939	12.8136	2.0232	6.7938	12.8134
3	**0.0000**	**0.0000**	**0.0000**	**0.0018**	**0.0065**	**0.0126**
4	-0.0268	-0.0886	-0.1651	-0.0268	-0.0887	-0.1652
5	30.6412	65.3778	81.4599	30.7920	65.5485	81.5169
6	8.5142	10.0719	9.8010	9.1109	10.4909	9.9308

7	8.3299	9.8253	9.5623	8.3707	10.0260	9.6617
8	-1.4662	-1.4161	-1.4177	-1.4474	-1.4073	-1.4084
9	-0.9500	-0.9161	-0.9173	-0.9569	-0.9134	-0.9156
10	0.0002	0.0002	0.0002	0.0006	0.0004	0.0003

3. 4. Deflection Check using Classic Structural Analysis

$$\delta = \frac{(q_{line\ load} + q_{self\ weight})l^4}{8EI} = 19.17\ mm \tag{23}$$

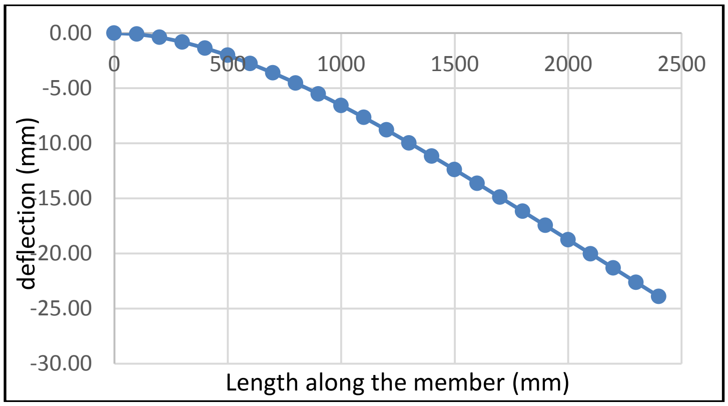

Fig. 6. Deflection in Vertical direction for Node 4 from GBT

Table. 5. Modal Contributions

Mode	W Displacement
1	0.000
2	**-19.183**
3	0.000
4	-4.540
5	-0.465
6	-0.027
7	-0.052
8	-0.037
9	-0.024
10	0.000
Total δ'	**-24.326**

It can be clearly seen from table 5 that second mode participation which is obtained from GBT analysis and classic structural analysis are exactly same. Fig. 6 shows the total vertical deflection of fourth node of the section which exactly matches the one obtained in Table 5.

3. 5. Individual Modal Participation for Different Lengths

Major advantage of GBT is we can get modal participation for varying lengths. This gives the user an intuition on how mode shape changes with varying lengths. Here, it is clearly seen in fig. 7, that global modes dominates as the length increases and the local and distortional effects are predominant for short and moderate length. It can also be observed that second and fourth (i.e. flexural and torsional) mode have an increasing participation with increase in length.

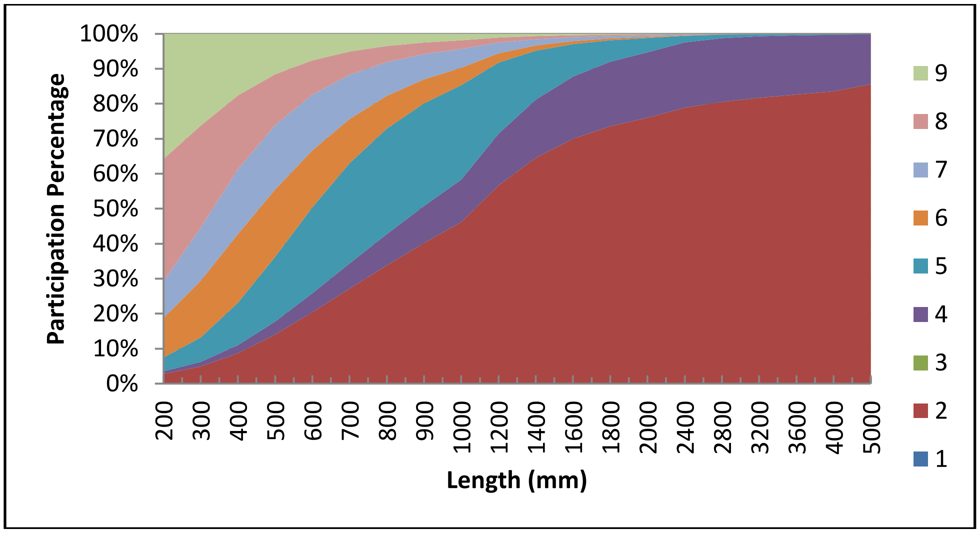

Fig. 7. Modal Participation graph for the example section

4. Conclusions

a) GBT analysis gives clear understanding of mode shapes that are involved during the deformation.

b) Coupled solution gives an edge over uncoupled solutions as it considers the effect of each mode on the other one.

c) In case of cantilever beams, as the length increases, global mode shapes play a major role whereas distortional mode shapes are dominant in short and medium lengths.

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