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Bayesian Statistics Applied to the Recovery of Copper and Cobalt-Bearing Ores: Establishment of the Probability Laws of Leaching Parameters

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Abstract **-** Currently, industries and researchers recognize the need for new ways to incorporate prior knowledge into processes. With the increasing accessibility of machine learning techniques through software packages, data-driven modeling applications are becoming more prevalent, incorporating more advanced techniques and analytics. Bayesian methods utilize advanced concepts of conditional probability for their inferences. This is crucial for developing robust and understandable solutions that can be used for prediction, optimization, or decision-making. In this paper, we discuss the implementation of Bayesian modeling in hydrometallurgy, focusing specifically on its application in the leaching process. Bayesian statistics provides the opportunity to incorporate prior knowledge into a model, making it a valuable approach in this field. Therefore, parameters that influence leaching, such as mineralogy, particle size, redox potential (pH), acidity, and oxido-reduction potential (ORP), were included as random variables in the model. This enables the designing of Bayesian networks with precisely defined a priori probability laws.

Keywords**:** Conditional probability, Bayesian statistics, modeling, hydrometallurgy, leaching, ore.

1. Introduction

The metallurgical industry has come to recognize in recent decades that it is more sensible and cost-effective to use historical data and available knowledge about ore processing and metal recovery processes to simulate a model, rather than concluding the future recovery of valuable metals based on specific operating conditions. This recognition has been driven by the advancements in artificial intelligence and machine learning technologies, which have been greatly influenced by the emergence of the Internet of Things (IoT), Big Data, and the fourth industrial revolution (4-IR) [1-3]. In a recent discussion, Sansana et al. [3] emphasized the importance for process engineering practitioners to incorporate techniques from fields such as computer science, applied mathematics, probability, and statistics.

One of the most emerging statistics in machine learning is Bayesian statistics, which utilizes the conditional probability approach and a specific type of inference based on Bayes' law. Whether in industrial, pilot, or laboratory conditions, the non-reproducibility of metallurgical results in most unit processes leads us to hypothesize that these results are the outcome of a random experiment. An experiment is considered random when, under similar conditions, the results exhibit variations. With the increasing complexity of modern technology in terms of extraction (such as ore richness and mineralogical variability) and growing environmental restrictions, predictive modeling of key process performance indicators (KPIs) in metallurgy becomes essential. In a multiple-input and multiple-output (MIMO) system, such as leaching, a conditional probabilistic

analysis approach can enhance the understanding of the variability of metallurgical results. One approach to reducing result variations would be to consider the input/output variables as random variables and utilize conditional probability to predict the metal dissolution process.

As machine learning techniques become more accessible, the use of data-driven models is expected to increase. These models are likely to incorporate more advanced analytics and become more common. Currently, there is a growing interest in techniques for modeling the dynamics of complex processes, as most existing applications assume that observations are independent. The availability of algorithms that automatically construct networks from databases, coupled with highly efficient inference algorithms, has made Bayesian networks a crucial model for calculating conditional probabilities. This is why Bayesian networks are widely used across various fields such as health and environment (gene localization, diagnosis, natural resource management), industry and transportation (automation and vehicle control), IT and networks (intelligent agents), marketing (data mining, customer relationship management), and management (decision support, financial analysis, risk management) [4-5].

On the other hand, its application in the field of ore processing and metal recovery is almost non-existent. However, there is a significant amount of literature suggesting that all the necessary ingredients for the application of this approach to metallurgical processes are available [4-6]. Despite the increasing popularity of Bayesian methods, there are only a few studies in the literature that utilize Bayesian approaches for metal recovery. However, the work of [6-12] can be cited as examples of using Bayesian networks as simulation tools enable experts to observe the behavior of their system in contexts that may be difficult for them to test themselves. This capability is commonly referred to as prediction [4].

Bayesian networks are a type of probabilistic analysis model that provides a mathematical framework and solid theoretical foundation for developing models of complex systems. These networks utilize a graphical representation to depict the relationships between input variables and the uncertainty associated with the data. By considering conditional dependencies and independence between random variables, the models offer a way to represent knowledge. Bayesian networks, which are a type of probabilistic graphical model, are used to perform tasks such as inference, parameter learning, and data structure identification. Moreover, these networks allow for incorporating prior probabilities on parameters or structure. When combined with machine learning technology, Bayesian networks enable the analysis of large datasets to extract valuable knowledge that can be used for decision-making, system control, predicting behavior, or diagnosing the causes of phenomena.

As mentioned earlier, Bayes' conditional probability has been applied to hydrometallurgical processes, although there are only a few reported works on this topic. However, the input variables have not yet been formally modeled as random variables. This research aims to bridge this gap. The objective of this paper is to discuss the implementation of input and output variables in hydrometallurgical processes for performing inference using the Bayesian learning approach. The paper will illustrate how this approach can be utilized to optimize a process and make predictions about the outcomes.

2. Conditional probability as applied to hydrometallurgy: State-of-the-art

This section presents the current state of the art regarding conditional probability from two perspectives: the key aspects of its development and its established applications in hydrometallurgy.

2.1. Development of Bayesian statistics

The advancements in probabilistic calculation in recent years have led to the development of probabilistic graphical MGP models [6]. The Bayesian approach, which experienced significant methodological development in the 20th century, has become a well-established framework for stochastic modeling, statistical inference, model selection, probabilistic forecasting, and decision support in an uncertain future [5-6]. Bayesian learning is now recognized as an emerging technique in various fields [6].

The success of probabilistic graphical modeling can be attributed to its ability to integrate knowledge and data from multiple sources into complex models, account for various sources of uncertainty, and quantify risks. For example, Cauvin [13] incorporated uncertainties and probability calculations in a study on soil and subsoil risks. Four categories of uncertainties were identified, and the Bayesian network was used to provide practical illustrations and treatment methods for each category. Mrad [14] further explored uncertain observations in a Bayesian network and their impact on beliefs about other variables. Three types of uncertain observations were identified. Additionally, Micallef et al. [15]

demonstrated how Bayesian analysis can address the uncertainty and variability associated with toxicokinetic data on human tissues.

Secondly, the practical implementation of Bayesian techniques has been facilitated by the development of calculation calculation resources and the availability of numerous Monte Carlo simulation tools resulting from advanced research in in probability theory. Several studies on the Markov chain Monte Carlo method (MCMC) have been reported, contributing contributing to the application of Bayesian statistics in various fields by considering uncertainty. For several decades, statisticians have widely used MCMC simulation methods to simulate complex multivariate distributions. Two significant procedures that revolutionized Bayesian inference are the Metropolis (-Hastings) algorithms and the Gibbs algorithm, enabling the estimation of realistic models. These methods have been applied in hydrometallurgical processes, including leaching.

In this context, Mbuya et al. [10] developed two Bayesian networks, with and without a posteriori knowledge, to graphically model leaching. These networks allowed for inference and prediction of the leaching yields of Cu and Co based on other input parameters. The dissolution of Co^{3+} from heterogenite requires reducing conditions, which can be achieved by using one or more reducing agents. These reducing agents can originate from added ferrous ions, iron present in valuable minerals such as chalcopyrite, or even gangue minerals like hematite. The conditional probability approach under the Bayes formalism was utilized to study the dissolution of Co^{3+} by considering other study parameters [8]. Mvita et al. [12] synergistically combined artificial neural networks and Bayesian networks to predict the dissolution behavior of Co^{3+} in sulfuric acid. Additionally, Saldana et al. [7] applied stochastic modeling, following the Bayesian approach, to study the heap leaching of copper ores using industrial data.

Finally, Bayesian reasoning lends itself well to graphical modeling which makes it possible to create expert systems. Bayesian networks were initially referred to as "probabilistic expert systems," where the graph was compared to the set of rules in a classical expert system, and the conditional probabilities were presented as a quantification of the uncertainty related to these rules. As the concept of expert systems has evolved [4], the Bayesian approach can adapt perfectly to this concept.

In short, the algorithms that automatically construct Bayesian networks from data sets, along with efficient inference algorithms, have made Bayesian networks a crucial model for calculating conditional probabilities. This is why Bayesian networks are widely used in various fields.

2.2. Application of Bayes' conditional probability to metallurgical processes

In a study conducted by Saldana et al. [7], the stochastic modeling of copper ore heap leaching was performed using a Bayesian network. The network was constructed with or without a priori knowledge, incorporating five parameters: leaching time, pile height, particle size, surface velocity of the leaching flow through the bed, and effective diffusivity of the solute within the pores of the particle, as well as the particle porosity. The developed Bayesian network, created using the GeNIe software, enabled the analysis of the conditional dependencies between the mentioned inputs and the recovery of copper, which served as the output in this study. The predictions made using the developed models yielded satisfactory results, as indicated by the good indicator values.

Koermer and Nobel [9] demonstrated how Bayesian methods can be applied to reconcile data from a rare earth solvent extraction pilot unit. Model selection methods for Bayesian models are already established and can be utilized to determine the most suitable model for a given data set, thereby enhancing data accuracy. The BayesMassBal package was specifically developed for real data applications. Mbuya et al. [8] employed a Bayesian approach to investigate the dissolution of cobalt-bearing minerals in a copper-cobalt mixed ore. Probabilistic models were used to predict the dissolution of cobalt-bearing minerals in a sulfuric acid medium with a reducing agent. The dissolution is primarily dependent on mineralogy, so $FeSO_4$ is used as a reducing agent for the trivalent form of cobalt [Co(Fe)OOH]. The presence of ferrous ions, which result from the dissolution of Fe-bearing minerals including Fe from Co(Fe)OOH, further enhances the dissolution of Co^{3+}. In their study, Mbuya et al. [10] examined the relationship between the random variables ($P(Fe^{2+}|T)$, $P(pH|T)$, $P(Fe^{2+}|pH)$) and the relationship between the random variables and the responses ($P(Co\text{-}yield|pH,Eh)$). This analysis enabled the construction of two Bayesian networks - one with prior knowledge and one without. The Bayesian network with prior knowledge was found to have the most appropriate {5-2} structure, as illustrated in Figure 1a generated using the XLSTAT software. Koermer and Nobel [16] employed a Bayesian statistical method to assess whether the flows entering and leaving a solvent extraction unit were steady. This method offers engineers a consistent metric that quantifies uncertainty. Therefore, understanding whether a process is in a steady state and the conditions associated with a steady state is crucial for valid techno-economic analysis.

Mbuya and Mulaba [11] proposed using the Bayesian approach for evaluating uncertainty in modeling. In leaching, uncertainty plays a crucial role at three levels that must be acknowledged : (1) In the leaching process itself, (2) In the modeling process, and (3) In the management of technological risks. Uncertainty primarily arises from the need to outcomes based on current data. Mvita et al. [12] conducted a comparison between artificial neural networks and probability (specifically Bayesian probability) in a scenario where multiple inputs (such as feed mineralogy, reducing agent, pH, and potential) were used to predict multiple outputs (such as dissolution recovery and lower dissolution rate). The study also explored the influence of prior knowledge on the presence and type of reducing agent, as expressed through conditional probability.

3. Methods

3.1. Bayes' theorem

The Bayesian point of view on learning is such that the learner has uncertainty about reality, which is expressed as a probability distribution in the space of models. This probability distribution is interpreted as a belief, not as the limit of a frequency. Any uncertainty is represented by a distribution. When we carefully analyze the system being studied, this allows us to revise the uncertainty in the model, generally by making this distribution more refined. In the case where the models considered are represented by a parameter X, the learning result is not a particular value for X but rather represents the models that we believe are possible.

Bayes' theorem makes it possible to connect the prior distribution P(X) and the posterior P(X | D):

$$P(X|D) = \frac{P(D|X)P(X)}{P(D)} \quad (1)$$

The quantity P(D | X) is called likelihood. Note that we are interested in this value and in P(X | D) simultaneously for all values of X, which poses certain computational challenges. The numerator, which equals P(D), is also called the evidence and is only used to normalize. It is therefore useless if the goal is only to find the most probable.

The conclusion of Bayesian inference is understood to be dependent on the prior distribution that has been chosen. If we start with a “bad” a priori, we obtain bad results. In particular, if the “real” generator model of D is not included in the model (i.e.⁄∃X such that P(D | X) is the real generator), then we cannot conclude anything.

On the other hand, there are advantages to Bayesian inference: it automatically takes into account the uncertainty due to the number of results in the confidence intervals it gives regarding the predictions made. In addition, it is theoretically safe from overfitting (we will automatically end up with a “simpler” decision function when we have few results). This is theoretical because it depends on the correctness of the a priori law, but in practice, it is a good way to avoid overfitting.

In summary: (1) We consider the quantity to be estimated (i.e. parameter a prior knowledge about X, expressed by a prior law P (X). (2) We assume that the law P (D|X) is known when X is known. (3) We then apply Bayes' theorem and the laws of probability for any inference.

The principle of Bayesian analysis makes it possible to update the information available on the plausible extent of the treatment effect values before the trial (expressed by a so-called a priori probability) with the data collected during the trial. (characterized by the likelihood function) to produce a revised estimate of the plausible range of treatment effect values (expressed by a so-called posterior probability). In this sense, the application of formula (1) to the dissolution of cobalt makes it possible to study the behavior of this metal in the presence of other elements whose influence would be important for its dissolution. Bayes' formula (1), very important in Bayesian statistics, is of great interest because it allows us to modify our knowledge of probabilities based on new information that can be implemented in the data set or in the probabilistic model through its variables.

3.2. Probabilistic graphical models

Launched in the early 1990s by Pearl [17], Bayesian networks were developed to account for uncertainties in expert and decision systems. Since then, their usage has continued to grow and they are well-suited for industrial applications, thanks to advancements in probabilistic calculation. Bayesian networks lie at the intersection of graph theory and probability theory. They provide a graphical representation of knowledge, showing direct dependence relationships, as

well as a probabilistic representation to handle uncertainties. These models offer a framework for representing knowledge by considering conditional dependencies and independence between random variables.

Bayesian learning, which is extremely popular in machine learning, is a mathematical modeling approach that combines combines statistics and artificial intelligence to represent knowledge and uncertainties. This is achieved using a structure called a Bayesian network (BN). A BN is a statistical analysis tool that uses a directed acyclic graph along with a probability table. It enables the analysis of large amounts of data to extract knowledge that is useful for decision-making, control, predicting system behavior, diagnosing causes of phenomena, and more. In essence, it is a decision-making tool that primarily uncovers causal relationships between variables.

The topology of a Bayesian network is shown in Figure 1a. This network is presented as a graphical model that shows the variables (called nodes) in a dataset and the probabilistic or conditional independencies that connect them. The causal relationship between nodes is représenter by links called arcs. The node (T), temperature, is considered the parent node. The node (Ag), the stirring speed, becomes the co-parent node, the nodes (pH or Fe^{2+}), hydrogen potential, or the concentration of ferrous ions, are considered child nodes. The arrows are called bows.

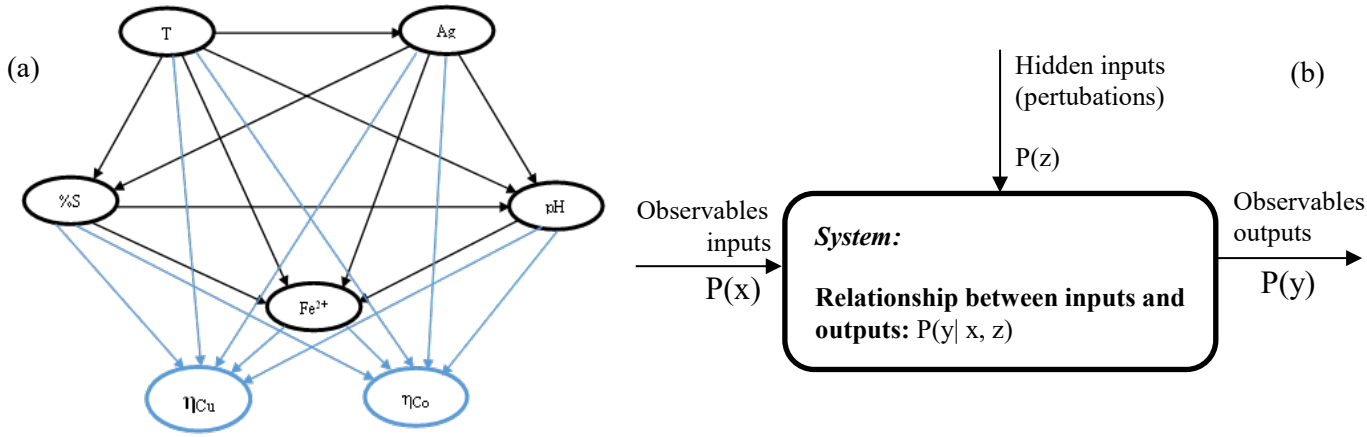

Fig. 1: Graphical representation of a Bayesian network illustrating the leaching process of a Cu-Co ore [10] (a) and Graphical representation illustrating a leaching system from a probabilistic perspective (b)

With such a network, the visualization of the causal relationships involved in the process studied is easy to understand and therefore, the context of the problem is better identified, which makes it possible to predict the consequences.

In Bayesian analysis, the explanatory (input variables) and explained (output variables) variables are modeled by a function whose probability distribution law is not known in advance. Such a characteristic corresponds well to random variables, so these are the random variables that are used in Bayesian conditional probabilistic modeling. Figure 1b schematizes the point of view adopted by stochastic methods (or probabilistic methods) for the study of systems (leaching in this case). Compared to Figure 1b, this model is completed by hypotheses on the probability distributions P(X) and P(Z) of the inputs. The input/output relationship is represented by a conditional distribution P(Y|X,Z), and the outputs are characterized by a probability distribution P(Y).

These probability distributions represent a certain level of knowledge of the physical system being modeled and include as special cases deterministic systems (systems in which a given state or initial condition will always produce the same results regardless of new information obtained).

Where in Figure 1b, the law P(Y|X,Z) takes into account the imperfect knowledge of the system (mineralogical variability, particle size variability, etc.), the P(Z) exploits our a priori knowledge (environmental conditions like the weather, etc.) as the hidden inputs, and the P(X) is for the observable inputs (pH, Temperature, Eh, particle size, etc.), including measurement errors. The conditional probability methods make it possible to calculate a probability P(Y) on each of the hydrometallurgical parameters or key performance indicators (KPIs) (like cobalt ions dissolution yield, acid consumption, etc.) given the above-mentioned information.

3.3. Random variables and probability law

In knowledge representation models based on a graphical description of random variables (Directed Acyclic Graph (DAG)), the nodes are the random variables and the edges are the (if possible) causal relationships between these (see Figure 1a).

A random variable is a function defined on a probability space that is compatible with the event algebras defined on its origin and destination spaces. It induces a law of probability on the destination space and the departure space. This is illustrated in Figure 2b. Given a probability space (a fundamental space Ω and probability measure P), we call random variable (r.v) on this space, any application X of Ω in R such that: X:ε(Ω)⟶F and ω⟼X(ω).

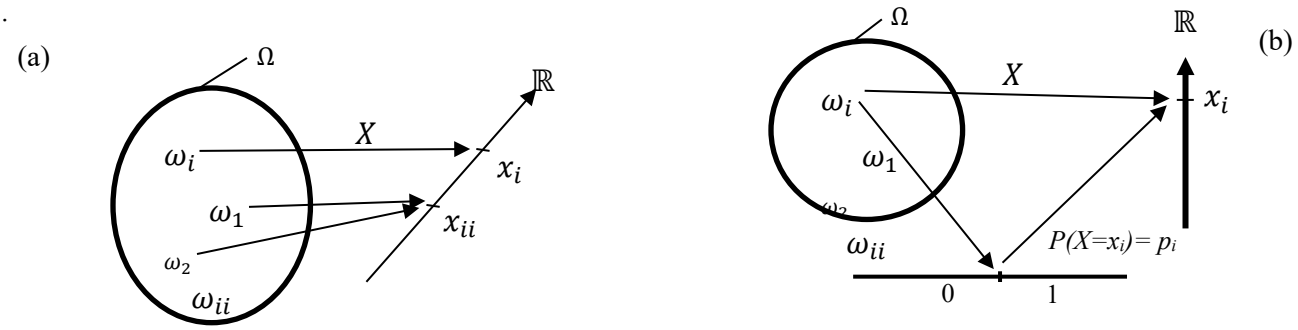

Fig. 2: Schematic representation of a random variable (a) and Probability Law of a Discrete Random Variable (b)

Each elementary event ω of Ω corresponds to a real number x associated with the random variable elementary events. The value x corresponds to the realization of the variable X for the elementary event.

A random variable is defined by the range of values it can take and the mathematical expression that describes the probability of each value occurring. This expression is known as the probability distribution of the random variable.

For a discrete random variable, the probability distribution is completely determined by the probabilities pi of the events $X=x_i$, where xi represents the possible values of X within the sample space X(Ω). The probability law of distribution can be represented as $(x_i,p_i)_i$, as shown in Figure 2b.

A random variable is considered discrete if it only takes on separate values within a given interval, whether that interval is bounded or unbounded. To simplify the writing, we will use the following notation throughout the course: P({$X=x_i$})) equivalent to P($X=x_i$) or p_i

4. Findings and discussions

In leaching, we aim to predict the performance of metallurgical indicators (such as dissolution yields of valuable metals, acid consumption, and the use of reducing or oxidizing agents). This involves understanding the potential outcomes of various choices, such as parameter settings and the selection of ores with their mineralogical variability. It is important to note that these outcomes can be influenced by random factors, both internal and external to the process.

A leaching test is considered a random experiment because when two tests are conducted under the same conditions, the results show some degree of variation. In many random phenomena, the results of an experiment can be expressed as a mathematical quantity, often represented by an integer or a real number. This mathematical representation is known as a random variable. The standard deviation values reported in several studies on the leaching of Cu-Co ores support this observation [18]. Even in confirmation or validation tests, where the results are expected to be known in advance, variations in the results are still observed.

The randomness of a leaching experiment is because some variables are initially random (such as mineralogy and particle size distribution) or become random during the process (such as redox potential, pH, and temperature). Figure 1a illustrates a probabilistic modeling of the leaching process, where the parameters are represented as random variables. A random variable is characterized by the range of values it can take and the mathematical expression of the probability associated with these values. This mathematical expression is referred to as the probability distribution law of the random variable.

4.1. Modeling mineralogy as a random variable in a leaching system

In a traditional approach to ore leaching, mineralogy is not considered as a variable. However, mineralogy should not not only be seen as a variable in the system being studied, but also as a random variable. This is because there is variability in mineralogy between different deposits and even within the same deposit. This variability justifies the different dissolution behaviors of metals, especially copper and cobalt. Additionally, the requirement for a feed with a cut-off grade that is compatible with metallurgical plant designs often means using a composite feed made up of ores from different sources.

In the case of leaching Cu-Co ores, a process called mineral-mineral leaching has been proposed. This process involves mixing oxidized and sulfide ores in the right proportions to achieve simultaneous dissolution of Cu and Co while minimizing the use of acid and reducing agents, or oxidants.

To incorporate mineralogy as a random variable in the Bayesian approach, it will be treated as a variable that provides prior knowledge of the probabilistic graphical model. For instance, when analyzing a sample of copper ore, mineralogical analysis can determine the types of minerals present and their proportions.

Let X be the random variable (the mineralogy in this case), and x_1, x_2, x_3,…, x_n the different values that the v.a will produce (minerals contained in the ore sample). The results of such mineralogical analysis can be presented in Table 2.

Table 2: Representation of the mineralogy of an ore sample as a random variable

Mineral types	x_i,	x_1	x_2	x_3	…	x_n
Proportions (%)	$r_i = f(x_i)$	r_1	r_2	r_3	…	r_n
	$P(X=x_i)$	p_1	p_2	p_3	…	p_n

With, x_1 = CuO, x_2 = Cu_2S, x_n = $CuCO_3.Cu(OH)_2$; we can also code them by the numbers 1 for x_1, 2 for x_2, 3 for x_3, …, n for x_n). Table 2 represents the mineralogy modeled as a random variable (lines 1 and 2). This is fully defined when we include line 3, which represents the probability distribution of the random variable. Using Table 2, we can calculate two crucial quantities: the expectation (E(X)) and the standard deviation (σ). These quantities provide a metallurgical interpretation of a metallurgical system using statistical results. The expectation of a random variable is the average of its values, taking into account their probabilities. The standard deviation measures the dispersion of the probability distribution of a random variable.

The significance of this reasoning is to incorporate prior knowledge into the studied network. For example, Saldana et al. [7] integrated the effective diffusibility of solutes in particle pores and grain porosity when using probabilistic graphical modeling for heap leaching.

4.2. Modeling particle size as a random variable in the leaching system

Particle size is the key factor that is closely related to mineralogy in leaching processes. Specifically, smaller particle sizes lead to a higher release of minerals. This release can be illustrated by a probability distribution law, which is represented by a cumulative pass or refusal curve normalized in the form of a Gaussian distribution. Consequently, particle size can be represented as a random variable. In this study, the particle size will be modeled based on the ASTM sieve series, using different values (x_i) as shown in Table 3.

Tableau 3. Representation of ore sample grain size as a random variable.

Fractions	x_i (µm)	(+45-53)	(+53-63)	(+63-75)	…	(+355-425)
Weight of retained	$r_i = f(x_i)$	r_1	r_2	r_3	…	r_n
Proportions of retained	$P(X=x_i)$	p_1	p_2	p_3	…	p_n

4.3. Modeling of hydrogen (pH) and redox (Eh) potentials as random variables in the leaching system.

As mentioned earlier, Bayesian reasoning offers the advantage of incorporating prior knowledge into a study. In the context of hydrometallurgy, where pH and Eh are two crucial thermodynamic quantities, Bayesian reasoning can incorporate the a priori understanding of metal dissolution based on the concept of gradient.

The same line of reasoning applies to both pH and Eh gradients. We know that metal dissolution is more rapid during the initial stages.

- In the case of leaching in a stirred tank reactor, we can consider the acidity gradient between two time points, t0 and t1. The difference (Δ) between the acidity at t_0 and t_1 gives us the acidity gradient. When operating at a constant pH, we will consider the volume of acid used to maintain the pH at its desired value for a specific duration, t_1.
- In the case of heap leaching, we consider the difference between the initial acidity and the acidity of the first percolating solution as the gradient.

To implement the acidity, we will follow the approach outlined in Tables 4 and 5.

Table 4: Representation of pH as a random variable

Acidity	x_i	x_1	x_2	x_3	...	x_n
Variation of acid volume	$r_i = f(x_i)$	$r_1=v_1$	$r_2=v_2$	$r_3=v_3$	...	$r_n=v_n$
Proportions of retained	$P(X=x_i)$	p_1	p_2	p_3	...	p_n

Table 5: Representation of acidity as a random variable

Acidity	x_i	x_1	x_2	x_3	...	x_n
Variation of acidity	$r_i = f(x_i)$	$r_1=\Delta_1$	$r_2=\Delta_2$	$r_3=\Delta_3$	...	$r_n=\Delta_n$
Proportions of retained	$P(X=x_i)$	p_1	p_2	p_3	...	p_n

Calculating the probability is straightforward and can be done using standard probability calculation methods. Therefore, pH (or Eh) is considered a random variable. Similar reasoning can be applied to model other parameters. Once the variables are fully defined (including modalities and the probability law), Bayesian networks can be constructed using the XLSTAT and GeNIe software. These software tools, used by Mbuya et al. [10] and Saldana et al. [7] respectively, allow for the optimization of the constructed networks and the necessary inferences.

Bayesian networks are concise representations of complex probability distributions. They have efficient inference algorithms that calculate posterior probability distributions for decision-making. Bayesian networks can also determine the instantiation, which refers to the values of all variables with the highest probability, known as "the most probable explanation". They can also determine the instantiation of a subset of variables with the highest probability.

4. Conclusion

This article demonstrated how conditional probability methods can be integrated into the new design and modern management of the metallurgical industry. It highlights one of the most emerging methods that address the methodological, technological, and operational challenges faced by process engineers. These solutions aim to incorporate prior and/or posterior knowledge about the processes, leading to more robust and comprehensible solutions. These solutions are capable of achieving prediction, optimization, and aiding in decision-making. The article showcases the application of Bayesian methods to metallurgical processes. It outlines the following capabilities of Bayesian methods : (i) integration of prior knowledge by modeling input parameters as random variables with a probability distribution. (ii) generation of inferences using Bayesian networks constructed by experts. These networks can be optimized using specialized software. (iii) modeling that considers uncertainties in the data and developed models. (iv) resolution of other metallurgical problems, such as material balance, through the development of codes that can be implemented in existing platforms. The article suggests that implementing Bayesian methods in leaching provides a promising avenue for its development, similar to other successful artificial intelligence techniques like artificial neural networks, which have already made significant contributions to the field of metallurgy.

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