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Optimal Designs and Reliability Acceptance Sampling Plans for Accelerated Copula-Based Dependent Competing Risks Model

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Abstract This paper considers the determination of the optimal design and reliability acceptance sampling plans (RASPs) in the presence of dependent competing causes of failure. It is assumed that the items are subjected to a constant stress accelerated life test (CSALT) under a traditional Type-I censoring scheme. The potential failure time under each failure mode is assumed to follow a Weibull distribution. The dependence structure between the potential failure times is modeled using a Gumbel copula. Using a log-linear stress translation function, optimal design parameters are obtained by solving an optimization problem under budgetary constraints. In addition, optimal cost-based RASPs are developed under CSALTs for the specified producer's and consumer's risks.

***Keywords*:** Dependent Competing failures, Gumbel Copula, Weibull Distribution, Type-I censoring, Constant-stress accelerated life testing

1. Introduction

Life testing plays an important role in reliability studies. In life testing, units are put into operation to obtain failure-time information to estimate the reliability of the products. For highly reliable products, the mean time to failure under normal operating conditions is often prohibitively long. Thus, accelerated life testing (ALT) was developed to obtain failure time information in a shorter duration to estimate the reliability of products in a reasonable time frame. In that life-testing scenario, the items are put on higher than usual stress levels to obtain failure-time information more quickly. A statistical model is used to fit the data at accelerated levels and then extrapolated to normal operating conditions. See Nelson [1], Bagdonavicius and Nikulin [2] for a detailed review on ALT.

Among various mechanisms to perform such accelerated tests, CSALT experiments are very popular among reliability practitioners. In CSALT experiment involving k stress levels, a sample of size n is taken from a lot and the sample is divided into k subgroups. For each subgroup, one pre-specified stress level is assigned and the life test is performed to obtain failure time information under the specified stress level. However, due to cost, time, and other resource constraints, censored life tests are used to collect lifetime information at each stress level. Type I and Type II are the two most common types of censoring schemes used in life-testing experiments. If the life test is stopped at a prefixed time τ, then it is known as Type-I censoring scheme, while if it is stopped after a fixed number of failures $r \leq n$, then it is known as Type-II censoring.

There are a number of works on inferences for the lifetime distribution under ALT with different censoring schemes. For example, statistical inference of generalized gamma distribution under type-I censoring was studied by Fan and Yu [3]. Wang et. al [4] studied inference under progressive type-II censoring for the Weibull distribution. Lin et. al [5] studied under type-I hybrid censoring for the lifetime distribution of the log-location scale family. The optimal design of four levels of CSALT was studied by Yang [6]. Tang et al. [7] presented two alternative ways of planning CSALTs for the Weibull distribution. Guan et. al [8] considered the generalized exponential distribution.

These works are considered for the single-failure mode items. However, due to the complex structure of modern products, the product may fail due to more than one failure mode. For example, in an automobile, the item fails due to a surface defect, interior defect, or both. Now, for exploring the failure mechanism of the product, we observe the time to failure along with the cause of failure. Such a model is known as the competing risk model, which is widely used in reliability studies. There are many works on the analysis of competing risk data for different distributions under ALTs. For instance, Klein and Basu [9] applied CSALT for exponential distribution with competing risk under different censoring schemes. Pascual [10] considered the problem of planning CSALT for known shape parameter Weibull distribution with competing risk setup. Later, he discussed the problem of Weibull distribution for unknown shape parameters and log-normal distribution [11].

The main assumption of the above works is that all competing failure modes are independent. However, in practice, the failure modes may not be independent. The copula approach is one of the most popular methods for modeling dependent competing risks. Using the copula method, the dependence structure between dependent competing failure modes can be described. Xu and Tang [12] used a coupla approach for dependent competing risk models under exponential distributions in CSALT. Wu et. al [13] studied for the Weibull distribution under progressive hybrid censoring.

To the best of our knowledge, there has been no work on optimal ALT and RASP for dependent competing risk models under type-I censoring. In this work, we consider optimal design and reliability acceptance sampling plans with dependent multiple failure mode items under multiple constant-stress ALTs. Also, budget is an important consideration in conducting such a life test. Thus, we need to determine the optimal allocation of test units under different stress levels and inspection times τ under a pre-fixed experimental budget. We also discuss the development of RASPs by minimizing the total experimental cost.

The rest of the paper is organized as follows. The model & assumptions are discussed in Section 2. Determination of optimal design and RASP are discussed in Section 3. Numerical results for both optimal design and RASP are discussed in Section 4. The conclusion and future studies are discussed in Section 5.

2. The Model & Assumptions

Let n identical test units be selected from the lot and put on an ALT with k elevated stress levels. Also, it is assumed that there are two competing causes of failure. At each stress level, the distribution of the item is derived in Section 2.1. After that, we present the background details along with the Fisher information matrix (FIM) in Section 2.2.

2.1. Distribution of lifetime

Let X_{ij} be the lifetime of the item due to j^{th} cause of failure at the stress level s_i. We assume that X_{ij} follows the Weibull distribution with cumulative distribution function (CDF),

$$F_{ij}(x) = 1 - \exp\left[-\left(\frac{t}{\lambda_{ij}}\right)^{\eta_j}\right]$$

and probability density function (PDF),

$$f_{ij}(x) = \frac{\eta_j}{\lambda_{ij}}\left(\frac{t}{\lambda_{ij}}\right)^{\eta_j-1} \exp\left[-\left(\frac{t}{\lambda_{ij}}\right)^{\eta_j}\right]$$

where $\lambda_{ij} > 0$ is the scale parameter and $\eta_j > 0$ is a shape parameter, for $i = 1,2 \ldots, k$ and $j = 1,2$. Now, we construct the reliability function of the items at any stress level s_i. As discussed earlier, the failure modes of the item are dependent, and the dependent structure is described by using the copula method.

According to Sklar's theorem, the joint distribution function F_Y of Y_1 and Y_2 can be expressed in terms of its marginal distribution F_{Y_1} and F_{Y_2} and a copula function $C(.,.)$ such that

$$F_Y(y_1, y_2) = \Pr(Y_1 \leq y_1, Y_2 \leq y_2) = C(F_{Y_1}(y_1), F_{Y_2}(y_2)) \quad y_1 \geq 0, y_2 \geq 0$$

Likewise, the joint reliability function R_Y of Y_1 and Y_2 can be expressed in terms of its marginal reliability function and R_{Y_2} and a copula function $\bar{C}(.,.)$ such that

$$R_Y(y_1, y_2) = \Pr(Y_1 > y_1, Y_2 > y_2) = \bar{C}\left(R_{Y_1}(y_1), R_{Y_2}(y_2)\right) \quad y_1 \geq 0, y_2 \geq 0$$

For simplicity, the Gumbel copula is considered. Then, the joint reliability function of Y_1 and Y_2 can be expressed as

$$R_Y(y_1, y_2) = \bar{C}\left(R_{Y_1}(y_1), R_{Y_2}(y_2)\right) = \exp\left[-\left\{\left(-\ln R_{Y_1}(y_1)\right)^{\theta} + \left(-\ln R_{Y_2}(y_2)\right)^{\theta}\right\}^{1/\theta}\right]$$

For this copula, the measure of association, Kendall's tau, is

$$\rho = 4\int_0^1\int_0^1 C(u,v)\, dC(u,v) - 1 = \frac{\theta-1}{\theta} \tag{1}$$

where $u = F_1(y_1)$ and $v = F_2(y_2)$ (For more details, see Nelson [14]).

Let X_i be the lifetime of the item at the stress level s_i. Now, using Gumbel copula, the reliability function of the item at the stress level s_i can be written as

$$R_i(t) = \bar{C}\big(R_{i1}(t), R_{i2}(t)\big) = \exp\left[-\left\{\left(\frac{t}{\lambda_{i1}}\right)^{\theta} + \left(\frac{t}{\lambda_{i2}}\right)^{\theta}\right\}^{1/\theta}\right]$$

The scale parameter λ_{ij} is related to the (possibly modified) stress level s_i with a log-linear stress-translation function, i.e, we have

$$\lambda_{ij} = a_{1j} + a_{2j}s_i$$

As an example, when voltage is used as the applied stress, $s_i = ln\, E_i$; whereas for temperature, $s_i = 1/T_i$. For convenience, the stress level is often standardized. Let ξ_i denote the standardized stress variable given by

$$\xi_i = \frac{s_i - s_0}{s_k - s_0}, \quad \text{for } i = 0, \ldots, k, \tag{2}$$

where s_0 is the level of stress at normal use condition and s_k is the highest level of stress. Therefore, $0 = \xi_0 < \xi_1 < \cdots < \xi_k = 1$. Thus, the scale parameters are connected in a log-linear relationship to the standard stress level ξ_i. The relation for the component j at stress level ξ_i to the parameter λ_{ij} is taken as

$$\lambda_{ij} = a_{1j} + a_{2j}s_i \text{ for } i = 1, \ldots, k.$$

Let C_i denote the indicator variable for the cause of failure, $C_i = j$ means the product fails due to cause j. The joint PDF of (X_i, C_i) at the stress level s_i is given by

$$f_{(X_i,C_i)}(t,j) = \frac{\partial \bar{C}\big(R_1(t), R_2(t)\big)}{\partial R_j(t)} f_{ij}(t) = \left[\left(\frac{t}{\lambda_{i1}}\right)^{\theta} + \left(\frac{t}{\lambda_{i2}}\right)^{\theta}\right]^{\frac{1}{\theta}-1} \left(\frac{t}{\lambda_{ij}}\right)^{\eta_j\theta-1} \frac{\eta_j}{\lambda_{ij}} \exp\left[-\left\{\left(\frac{t}{\lambda_{i1}}\right)^{\theta} + \left(\frac{t}{\lambda_{i2}}\right)^{\theta}\right\}^{1/\theta}\right]$$

2.2. Likelihood function

Consider a random sample of size n put on a life-testing experiment under the Type-I censoring scheme in a CSALT experiment. Therefore, n units are divided into a k subgroup with size $n_1, \ldots, n_k$, where $n_1 + \cdots + n_k = n$. We assume that n_i units is put on a life test under stress s_i, $i = 1, \ldots, k$. The test is terminated after a pre-determined time point τ is reached. Let d_{ij} be the number of failures due to cause j under stress level s_i. Let the observed lifetime data set due to the cause j are $\mathrm{D}_{ij} = \{t_{ij1}, t_{ij2}, \ldots, t_{ijd_{ij}}\}$. The observed data set under the stress level s_i is $\mathrm{D}_i = \{\mathrm{D}_{i1}, \mathrm{D}_{i2}\}$ and total observed data under this life-testing experiment is $\mathrm{D} = \{\mathrm{D}_1, \mathrm{D}_2, \ldots, \mathrm{D}_k\}$. We assume that the vector of parameters is $\boldsymbol{\psi} = (b_{11}, b_{12}, b_{21}, b_{22}, \eta_1, \eta_2, \theta)$. For the observed data D_i at the stress levels_i, the likelihood function (l.f.) can be written as

$$L_i(\boldsymbol{\psi} \mid \mathrm{D}_i) = \left[\left(\frac{t_{ijk}}{\lambda_{i1}}\right)^{\theta} + \left(\frac{t_{ijk}}{\lambda_{i2}}\right)^{\theta}\right]^{\frac{1}{\theta}-1} \left(\frac{t_{ijk}}{\lambda_{ij}}\right)^{\eta_j\theta-1} \frac{\eta_j}{\lambda_{ij}} \exp\left[-\left\{\left(\frac{t_{ijk}}{\lambda_{i1}}\right)^{\theta} + \left(\frac{t_{ijk}}{\lambda_{i2}}\right)^{\theta}\right\}^{1/\theta}\right]$$

The log l.f. at the stress level s_i is given by

$$l_i(\boldsymbol{\psi} \mid \mathrm{D}_i) = ln\, L_i(\boldsymbol{\psi} \mid \mathrm{D}_i)$$

The total l.f. can be written as

$$L(\boldsymbol{\psi} \mid \mathrm{D}) = \prod_{i=1}^{k} ln\, L_i(\boldsymbol{\psi} \mid \mathrm{D}_i)$$

The total log l.f. is

$$l(\boldsymbol{\psi} \mid \mathrm{D}) = ln\, L(\boldsymbol{\psi} \mid \mathrm{D}) = \sum_{i=1}^{k} l_i(\boldsymbol{\psi} \mid \mathrm{D}_i)$$

Let $I(\boldsymbol{\psi})$ denotes the FIM. By definition, we have

$$I(\boldsymbol{\psi}) = E\left[-\frac{\partial^2 l(\boldsymbol{\psi} \mid \mathrm{D})}{\partial\boldsymbol{\psi}\partial\boldsymbol{\psi}'}\right] = E\left[-\sum_{i=1}^{k}\frac{\partial^2 l_i(\boldsymbol{\psi} \mid \mathrm{D}_i)}{\partial\boldsymbol{\psi}\partial\boldsymbol{\psi}'}\right] = \sum_{i=1}^{k} I_i(\boldsymbol{\psi}) = \sum_{i=1}^{k} n\pi_i \mathrm{F}_i(\boldsymbol{\psi})$$

where $\mathrm{F}_i(\boldsymbol{\psi})$ is the FIM for one item at the stress level ξ_i, for $i = 1, \dots, k$, $\boldsymbol{\psi}'$ is the transpose of $\boldsymbol{\psi}$ and $\pi_i = n_i/n$, for $i = 1, \dots, k$, i.e., π_i is the proportion of samples put in the life test at the stress level ξ_i. Clearly, $0 \le \pi_i \le 1$. The decision variable vector under this life testing plan is denoted by $\boldsymbol{\zeta} = (n, \boldsymbol{\pi}, \boldsymbol{\xi}, \tau)$, where $\boldsymbol{\pi} = (\pi_1, \dots, \pi_k)$ with $\pi_1 + \cdots + \pi_k = 1$and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{k-1}, 1)$.

3. Optimal design & RASP

Here, we propose the optimal design and RASP setup.

3.1. Optimal design

Several design criteria have been proposed in the context of finding the optimal life testing plan. In this paper, we find the optimal design parameter $\boldsymbol{\zeta}$ by minimizing the variance of the system reliability function at a pre-specified time point t_0 under a certain budget constraint C_B. Let $C(\boldsymbol{\zeta})$ denote the TC of the life test experiment. The TC of the experiment can be taken as

$$C(\boldsymbol{\zeta}) = nC_s + C_d E[D] + C_t\tau$$

where C_s is the cost per item, C_d is the cost per failure item, and C_t is the cost per unit of time. $E[D]$ denotes the expected number of failures during the life test and can be defined as $E[D] = \sum_{i=1}^{k} n_i R_i(\tau)$. Therefore, our problem is as follows:

minimize $\phi(\boldsymbol{\zeta})$ =var($R(t_0)$)

subjected to

$C(\boldsymbol{\zeta}) \le C_B$

$0 \le \pi_i, \xi_i \le 1$, $i = 1, \dots, k, n$is positive a integer value and $\tau > 0$

where $\mathrm{var}(R(t_0)) = \nabla_{\boldsymbol{\psi}} R(t_0)'[I(\boldsymbol{\psi})]^{-1}\nabla_{\boldsymbol{\psi}} R(t_0)$, where $\nabla_{\boldsymbol{\psi}} R(t_0) = \left(\frac{\partial R(t_0)}{\partial b_{11}}, \frac{\partial R(t_0)}{\partial b_{12}}, 0, 0, \frac{\partial R(t_0)}{\partial \eta_1}\frac{\partial R(t_0)}{\partial \eta_2}\frac{\partial R(t_0)}{\partial \theta}\right)$. Solving the above problem, we get the optimal designing parameter $\boldsymbol{\zeta}^*$. Now, we provide upper bounds of n, τ. Let n_B and τ_B be the upper bound of n, τ respectively. $\boldsymbol{\zeta}^* = (n^*, \pi^*, \xi^*, \tau^*)$ be the optimal sampling parameters. Since all costs C_s, C_d and C_t are positive, we get,

$$C_B \ge C(\boldsymbol{\zeta}^*) \ge n^* C_s + C_t\tau^* \ge n^* C_s \tag{3}$$

From (3), we get $n^* \le \frac{C_B}{C_s}$. Therefore $n_B = \left\lfloor \frac{C_B}{C_s} \right\rfloor$, where $\lfloor y \rfloor$ is the greatest integer less than or equal to y. Also, from (3), we get $\tau^* \le \frac{C_B - C_s n^*}{C_t}$. Therefore $\tau_B = \frac{C_B - C_s n^*}{C_t}$.

3.2. Sampling plan

Consider that a sample of size n is selected from a lot. The items are divided into k pre-specified subgroups with the proportion $\pi_1, \dots, \pi_k$, where $\sum_{i=1}^{k} \pi_i = 1$. The lifetime quantile t_p corresponding to a given fraction of nonconforming item p under normal conditions satisfies the following equation:

$$1 - p = \exp\left[-\left\{\left(\frac{t}{\lambda_{i1}}\right)^{\theta} + \left(\frac{t}{\lambda_{i2}}\right)^{\theta}\right\}^{1/\theta}\right] \tag{4}$$

The consumer accepts the lot when the reliability of the product at a specified time t_p is greater than a specified required reliability R_c. Let $\hat{R}(t_p)$ be the estimated reliability at normal conditions. Therefore, the lot is accepted if $\hat{R}(t_p) > R_c$ and the lot is rejected $\hat{R}(t_p) \ge R_c$.

In order to design RASP for CSALT under the type-I censoring scheme, we need to derive the sample size n, the proportion of the sample π_i at each stress level s_i, for $i = 1, \dots, k$, the time duration τ and R_c. First, we derive the expression of n and R_c such that the plan meets the producer's risk α and consumer's risk β. According to a mutual agreement between the producer and the consumer, the lot is considered acceptable if $p < p_\alpha$ and the lot is considered rejectable if $p > p_\beta$. Let t_{p_α} and t_{p_β} be the lifetime quantiles corresponding to given nonconforming item p_α and p_β, respectively. Using equation (4), the lot is accepted when $t_\mathrm{p} > t_{p_\alpha}$ and rejected when $t_p \le t_{p_\beta}$. Now, the producer's risk α is defined as the probability of a good lot being mistakenly rejected i.e,

$$1-\alpha = \Pr\left[\hat{R}(t_p) > R_c \mid t_p = t_{p_\alpha}\right]$$

and the consumer's risk β is defined as the probability of a bad lot being mistakenly accepted, i.e,

$$\beta = \Pr\left[\hat{R}(t_p) > R_c \mid t_p = t_{p_\beta}\right],$$

where $\hat{R}(t_p) = \exp\left[-\left\{\left(\frac{t_p}{\hat{\lambda}_{01}}\right)^{\hat{\eta}_1\hat{\theta}} + \left(\frac{t_p}{\hat{\lambda}_{02}}\right)^{\hat{\eta}_2\hat{\theta}}\right\}^{\frac{1}{\hat{\theta}}}\right]$, $\hat{\eta}_1$, $\hat{\eta}_2$ and $\hat{\theta}$ be the MLEs of η_1, η_2 and θ respectively, $\hat{\lambda}_{01} = exp\,(\hat{b}_{11})$ and $\hat{\lambda}_{02} = exp\,(\hat{b}_{12})$are the MLEs of λ_{01} and λ_{02} respectively. Therefore, $\hat{\boldsymbol{\psi}} = (\hat{b}_{11}, \hat{b}_{12}, \hat{b}_{21}, \hat{b}_{22}, \hat{\eta}_1, \hat{\eta}_2, \hat{\theta})$ is the MLE of $\boldsymbol{\psi}$. The asymptotic distribution of $\hat{\boldsymbol{\psi}}$ follows multivariate normal distribution with mean ψ and variance-covariance matrix $I^{-1}(\psi)$. By using the delta method, $\hat{R}(t_p)$ follows normal distribution with mean $R(t_p)$ and variance $V = \mathrm{var}(R(t_p))$.

Let R_α and $R_\beta$be the values of $R(t_p)$, and V_α and $V_\beta$be the values of V when $t_p = t_{p_\alpha}$and $t_p = t_{p_\beta}$, respectively. From the asymptotic property of MLE, we get

$$\Pr\left[Z > \frac{R_c - R_\alpha}{\sqrt{V_\alpha}}\right] = 1-\alpha$$

and

$$\Pr\left[Z > \frac{R_c - R_\beta}{\sqrt{V_\beta}}\right] = \beta$$

From the above two equations, we obtain

$$R_c = \frac{R_\alpha z_{1-\alpha}\sqrt{V_\alpha} - R_\beta z_\beta \sqrt{V_\beta}}{z_{1-\alpha}\sqrt{V_\alpha} - z_\beta\sqrt{V_\beta}} \tag{5}$$

and

$$n = \left(\frac{z_{1-\alpha}\sqrt{V_\alpha} - z_\beta\sqrt{V_\beta}}{R_\alpha - R_\beta}\right)^2$$

where z_γ is the γ upper percentile of a standard normal distribution.

For given α, β, p_α and p_β, a constraint-optimization problem can be written as

Minimize $C(\boldsymbol{\zeta})$

subject to

$$n = \left(\frac{z_{1-\alpha}\sqrt{V_\alpha} - z_\beta\sqrt{V_\beta}}{R_\alpha - R_\beta}\right)^2$$

$0 \le \pi_i, \xi_i \le 1,\ i = 1, \ldots, k, n$ is positive a integer value and $\tau > 0$

Solving the above problem, we get the optimal designing parameter $\boldsymbol{\zeta}^*$ and using equation (5), we get the acceptance limit R_c.

4. Numerical Results

Here we consider Nelson's [1] example of a Class-B motor insulation system having three failure modes, namely, turn, phase and ground. In that example, 40 items are put in an ALT experiment where the temperature is the only stress variable. Forty items are divided into 4 subgroups with equal sizes. Therefore 10 items are put on a life test under temperatures 150°C, 170°C, 190°C and 220°C. It is observed that the number of breakdowns due to phase is negligible. Therefore, we consider only two failure modes, turn and ground. The data is divided by 100 and fits into the Weibull distribution. For $\theta = 1$, the MLE are $\hat{b}_{11} = 4.458$, $\hat{b}_{12} = 4.255$, $\hat{b}_{21} = -4.881$, $\hat{b}_{22} = -4.514$, $\hat{\eta}_1 = 5.445$ and $\hat{\eta}_2 = 2.432$.

In this article, it is assumed that items are having only two failure modes, turn and ground put in a temperature-accelerated life testing experiment. The usage and maximum allowable temperature are assumed to 130°C and 220°C respectively. We consider different values of ρ to represent different dependence structures of turn and ground failure modes. When $\rho = 0,1/2,2/3$, the copula parameter $\theta = 1,2,3$ using equation (1), where $\rho = 0$ represents the independent case. From equation (2), we can write, $\frac{1}{T_i} = \frac{1}{130} + \xi_i \times \left(\frac{1}{230} - \frac{1}{130}\right)$

Example 1: The cost components are taken as $C_s = 0.1$, $C_d = 0.1$, $C_t = 0.5$ and $C_B = 10$. The variance of the reliability function is calculated at $t_0 = 30$ under normal operating conditions. The optimal sample size n^*, optimal inspection time, τ^* and for low-stress level, optimal proportion of the sample, $\pi_1{}^*$and optimal standardized stress level ${\xi_1}^*$ are provided in Table 1. For high-stress levels, the optimal proportion of sample ${\pi_2}^* = 1 - {\pi_1}^*$ and the optimal standardized stress level ${\xi_2}^* = 1$ by definition. The variance of the reliability function at optimal design parameters $\boldsymbol{\zeta}^* = (n^*, \boldsymbol{\pi}^*, \boldsymbol{\xi}^*, \tau^*)$, where $\boldsymbol{\pi}^* = ({\pi_1}^*, 1 - {\pi_1}^*)$ and $\boldsymbol{\xi}^* = ({\xi_1}^*, 1)$ are given in Table 1. Also, the optimal sample size ${n_1}^* = n^* {p_1}^*$ and optimal temperature ${T_1}^*$ at low stress level are also given in Table 1. For high-stress level, the temperature is always $230°\,C$ and the optimal sample size is ${n_2}^* = n^* - {n_1}^*$.

Table 1: Optimal design for different values of θ

θ	n^*	${\pi_1}^*$	${\xi_1}^*$	τ^*	${n_1}^*$	$T_1(°C)$	$\phi(\zeta^*) \times 10^2$
1	25	0.764	0.311	9.200	19	150.32	9.773
2	25	0.782	0.316	9.203	20	150.70	9.181
3	25	0.784	0.329	9.116	20	151.69	8.913

Next, we consider the effect of parameters on optimum solution. The effect of the parameters C_s, C_t and C_B are given in Tables 2, 3 and 4, respectively.

Table 2: Optimal design for different values of θ and C_s

C_s	θ	n^*	${\pi_1}^*$	${\xi_1}^*$	τ^*	${n_1}^*$	$T_1(°C)$	$\phi(\zeta^*) \times 10^2$
0.1	1	47	0.780	0.313	9.155	37	150.47	6.723
	2	47	0.780	0.313	9.155	37	150.47	6.723
	3	47	0.781	0.327	9.014	37	151.54	6.548
0.2	1	17	0.765	0.313	9.240	13	150.47	11.818
	2	17	0.783	0.316	9.251	13	150.70	11.104
	3	17	0.785	0.329	9.188	13	151.69	10.766

In Table 2, it is observed that the optimal parameters except n^* do not change with C_s. The optimal sample size decreases with C_s as expected. Therefore, the variance of the reliability function increases with C_s. However, in Tables 2 and 3, it is seen that all optimal parameters are affected by C_t and C_B. The optimal inspection time τ^* decreases with C_t. Due to prefixed experimental budget C_t, the upper bound of τ decreases. Therefore, it may be noted that τ^* decreases with C_t. Also, in short time for better inference, we need to conduct the life test test with higher stress. Therefore, ${\xi_1}^*$ increases with C_t. Also, when $C_t = 0.1$, it is observed that ${\xi_1}^* = 0$ and ${\pi_1}^* = 1$.

Table 3: Optimal design for different values of θ and C_t

C_t	θ	n^*	${\pi_1}^*$	${\xi_1}^*$	τ^*	${n_1}^*$	$T_1(°C)$	$\phi(\zeta^*) \times 10^2$
0.1	1	30	1	0	34.489	30	130	5.237
	2	30	1	0	34.489	30	130	5.167
	3	30	1	0	34.489	30	130	4.985
1	1	16	0.692	0.454	4.948	11	161.91	16.450
	2	16	0.716	0.456	4.959	11	162.14	15.341
	3	16	0.719	0.469	4.930	12	163.29	14.865

Table 4: Optimal design for different values of θ and C_B

C_B	θ	n^*	${\pi_1}^*$	${\xi_1}^*$	τ^*	${n_1}^*$	$T_1(°C)$	$\phi(\zeta^*) \times 10^2$
5	1	11	0.698	0.438	5.259	8	160.58	19.236
	2	12	0.713	0.459	4.485	9	162.41	17.932
	3	11	0.723	0.455	5.230	9	162.05	17.393
15	1	36	0.826	0.229	14.182	30	144.37	6.520
	2	39	0.829	0.234	13.121	32	144.72	6.363
	3	39	0.830	0.246	12.991	32	145.56	6.199

Example 2: Next we study the RASP for given $p_\alpha = 0.05$ and $p_\beta = 0.2$. The value of other components are same as in Example 1. Now for $\alpha = 0.05, 0.1$ and $\beta = 0.05, 0.1$, the optimal RASP are tabulated in Table 5. It is seen that the sample size decreases with both α, β. Also it is seen that the total experimental cost at optimal design decreases with θ. The acceptance limit R_c increases with θ.

Table 5: Optimal RASP for different values of θ and (α, β)

α	β	θ	n^*	π^*	ξ^*	τ^*	n_1^*	$T_1(°C)$	R_c	$TC(\boldsymbol{\zeta}^*)$
		1	142	0.916	0.080	28.312	130	134.68	0.868	45.908
		2	159	0.859	0.144	21.138	137	138.68	0.872	45.598
0.05	0.05	3	156	0.857	0.161	20.246	134	139.78	0.88	44.803
		1	126	0.846	0.164	18.561	107	139.98	0.876	37.039
		2	127	0.829	0.187	17.11	105	141.5	0.881	36.481
0.05	0.1	3	125	0.829	0.204	16.452	103	142.65	0.889	35.757
		1	134	0.862	0.143	20.991	116	138.61	0.858	40.124
		2	136	0.845	0.164	19.488	115	139.98	0.862	39.744
0.1	0.05	3	134	0.845	0.181	18.737	113	141.1	0.871	39.053
		1	106	0.825	0.190	16.659	87	141.7	0.867	31.715
		2	106	0.825	0.190	16.659	87	141.7	0.872	31.715
0.1	0.1	3	104	0.815	0.227	15.012	85	144.23	0.88	30.592

5. Conclusion

This work considered the determination of the optimal design of the life testing experiment for dependent competing risk data when the items are put on CSALT under type-I censoring. We have also considered the determination of RASP under this setup. In this paper, we have considered Weibull distribution. However, the proposed method can be applied for other lifetime distributions. Also, in this work, we considered Gumbel copula to model the dependency of the competing risk data. We can use other Archimedean copulas like Clayton copula, Pareto copula, etc. Also for simplicity, we considered competing risk data with two failure modes. It can be extended to more than two failure modes (see [15]).

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