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Linear Prediction for Stationary Random Fields with an Application to Poverty Levels in Texas

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Abstract - This paper addresses the linear prediction problem for stationary random fields with a nonsymmetrical half-plane past (NSHP) framework. We establish an explicit autoregressive series representation for the multi-step ahead linear predictor and derive the necessary and sufficient conditions for its mean square convergence. Additionally, recursive relations between prediction coefficients for the infinite past predictor are presented, facilitating explicit computation of multi-step coefficients. The practical applicability of the proposed methodology is demonstrated through an analysis of an academic example and a lattice dataset representing poverty levels across Texas counties, modelled using a 2D spatial autoregressive model.

***Keywords*:** Stationary random fields, autoregressive models, multi-step ahead prediction, Lattice Data, Poverty level

1 INTRODUCTION

Spatial prediction methods have seen significant advancements due to high demand in fields like agriculture, climatology, ecology, econometrics, geology, medical studies, oil prospecting, and water pollution analysis, where data is typically available at specific locations [14]. A widely used approach is kriging, a geostatistical linear least squares estimation method for predicting natural phenomena at unsampled locations as a weighted average of nearby observations. Determining the weights requires specifying a parametric covariance model, which depends on the covariances between observed and interpolated locations. A critical step involves fitting the experimental variogram with a suitable parametric model, as described in [1]. However, selecting a covariance model is often subjective [2] and may result in biased predictions if the model is mis specified. Nonparametric methods for estimating covariance functions have been explored [3,4], but they remain challenging, particularly when the assumption of uniform dependence across points does not hold for irregular data. Thus, developing prediction methods with minimal assumptions on the covariance structure remains a significant challenge in spatial statistics.

In this paper, we address the prediction of stationary random fields using nonsymmetrical half-plane pasts. Specifically, we consider the structure of random processes over $S \subseteq \mathbb{Z}^2$ (2-D discrete random fields) and formulate the 2-D linear prediction problem analogous to the Wiener-Kolmogorov framework for 1-D processes. Unlike the 1-D case, where the notions of "past" and "future" are naturally defined, 2-D random fields lack a universal ordering, requiring a specific definition of "past" and "future". For instance [6,7] defined the predictor support as a nonsymmetrical half-plane (NSHP) and derived a Wold-type decomposition for stationary random fields. They also developed the associated spectral theory. Building on this, [5] demonstrated that multiple total-order and NSHP definitions lead to a countably infinite Wold-type decomposition. In [8], we investigated linear prediction for stationary random fields with NSHP and established an explicit autoregressive series representation for the best multi-step ahead predictor. We also provided necessary and sufficient conditions for the mean square convergence of these series.

2 AUTOREGRESSIVE REPRESENTATION OF THE MULTI-STEP AHEAD LINEAR PREDICTOR

The multi-step ahead prediction problem of stationary random fields has been studied by when the third quadrant is used as the past. We extend their pioneering work to random fields with nonsymmetrical half-plane past. Let $X(s,t);(s,t)\in\mathbb{Z}^2$ be a PND stationary random field. The totally ordered NSHP support is a favorable type of support in the sense that it yields a natural extension to the 1-D results; this fact has been shown in [6,7]. In what follows, we assume all definitions and theorems are stated with respect to the total order and nonsymmetrical half-plane (NSHP); and the 2-D random field $X(s,t)$. First, we introduce some basic definitions related to the NSHP prediction scheme. Next, we state the 2-D Wold-type orthogonal decomposition theorem.

Definition 1 ([6]). *We call a nonsymmetrical half-plane past (NSHP) any subset S of $\mathbb{Z}^2$ satisfying S stable under addition; $S\cup -S=\mathbb{Z}^2$; $S\cap -S=(0,0)$.*

The NSHP support, S, results from the total-order definition (1a). The total order is defined as a raster-scan: top-to-bottom column-after-column from left to right. In mathematical terms:

$$(s_1,t_1)<(s_2,t_2)\ \ iff(s_1,t_1)\in\{(k,l)\,|\,k=s_2,l<t_2\}\cup\{(k,l)\,|\,k<s_2,-\infty<l<\infty\} \quad (1a)$$

and the order $\leq$ is naturally defined by

$$(s_1,t_1)\leq(s_2,t_2)\,iff(s_1,t_1)<(s_2,t_2)\ or\,(s_1,t_1)=(s_2,t_2) \quad (1b)$$

which coincides with the lexicographic order. Figure 1 illustrates this support graphically.

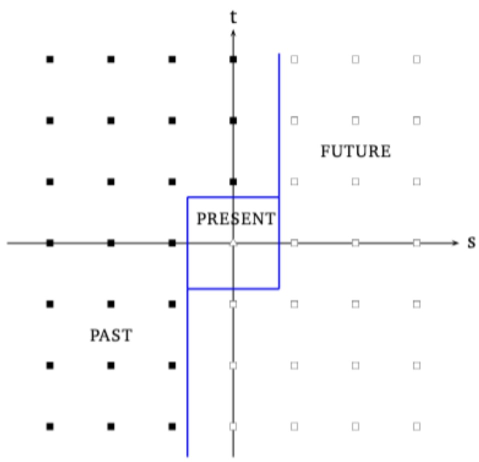

Figure 1. NSHP total order definition

The procedure for solving the $(h_1,h_2)-$step ahead linear prediction problem with respect to the total order and nonsymmerical half-plane (NSHP) support defined by (1a) involves the construction of predictor of future values as a linear combination of $X(k,l),(k,l)\in S$ which are close to $X(s+h_1,t+h_2)$, $(h_1,h_2)\geq(0,0)$ in the sense of mean squared error. The collection of all finite linear combinations of elements in the space and its closure are also included in the space. First, we fix our attention on the problem of finding convergent representation for the one-step ahead linear predictor $P_{H^S}X(s,t)$, i.e. the minimum norm linear causal and continuous support predictor of $X(s,t)$. We show that when (3) converges, such a representation exists.

Theorem 2 ([8]). *Let $X(T)$; $T\in\mathbb{Z}^2$ be a PND stationary random field. The one step ahead linear predictor $P_{H^S}X(s,t)$ of $X(s,t)$ possesses a convergent series representation given by*

$$P_{H^S}X(s,t) = -\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} a_{k,l}X(s-k,t-l), (k,l) \neq (0,0) \quad (2)$$

if and only if $\varepsilon(s,t)$ *has the series representation*

$$\varepsilon(s,t) = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} a_{k,l}X(s-k,t-l) \qquad (3)$$

In the following theorem, we show that when (3) converges, we can find a convergent representation for all (h_1,h_2)-step ahead linear predictors, $(h_1,h_2) \geq (0,0)$, $(h_1,h_2) \neq (0,0)$.

Theorem 3 ([8]). *Let* $X(T)$; $T \in \mathbb{Z}^2$ *be a PND stationary random field. Then, for any* $(s+h_1, t+h_2) \in S^c$, *where* S^c *is the complement of* S. *The* $(h_1,h_2)-$*step ahead predictor of* $X(s+h_1, t+h_2)$, $(h_1,h_2) \geq (0,0)$, $(h_1,h_2) \neq (0,0)$ *based on the past* S *possesses a convergent series representation given by*

$$P_{H^S}X(s+h_1,t+h_2) = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} a_{k,l}^{(h_1,h_2)}X(s-k,t-l), (k,l) \neq (0,0) (4)$$

where

$$a_{k,l}^{(h_1,h_2)} = -\sum_{p=0}^{h_1}\sum_{q=0}^{h_2} b_{h_1-p,h_2-q}a_{p+k,q+l}, (k,l) \geq (0,1), (5)$$

if and only if $\varepsilon(s,t)$ *has the convergent series representation* (3).

According to the previous theorem, the best $(h_1,h_2)-$ step ahead predictor based on the infinite past S, chosen so that the prediction error is a white noise according to which the process $X(s,t)$ must be able to be represented, is

$$P_{H^S}X(s+h_1,t+h_2) = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} a_{k,l}^{(h_1,h_2)}X(s-k,t-l), (k,l) \neq (0,0), (h_1,h_2) \geq (0,0)$$

The following relations are analogous to the relations (2.2) in [12] in the one-dimensional case. From [5] we have

$$a_{k,l}^{(h_1,h_2)} = \sum_{p=0}^{h_1}\sum_{q=0}^{h_2} b_{p,q}a_{k+h_1-p,l+h_2-q} \quad (k,l) \geq (0,1), \ (6)$$

Step recursive relations when the prediction is based on infinite past can be immediately deduced from relations (5) and (6) as follows,

$$a_{k,l}^{(h_1,h_2)} = a_{k,l+1}^{(h_1,h_2-1)} + \sum_{p=0}^{h_1} b_{p,h_2}a_{k+h_1-p,l} \quad (k,l) \geq (0,1) \qquad (7)$$

We shall notice that $a_{k,l}^{(0,0)} = -a_{k,l}$ for all $(k,l) \geq (0,1)$. The parameters $a_{k,l}^{(h_1,h_2)}$, $(h_1,h_2) > (0,0)$, can be calculated recursively using (7).

3 NUMERICAL EXAMPLE

The proposed prediction methodology is validated through two case studies: an academic example and a lattice dataset modeling poverty level across Texas counties using a 2D spatial autoregressive model. These examples illustrate the model's ability to capture spatial dependencies and provide accurate predictions.

3.1 First Order Multiplicative Spatial Autoregressive Model (MSAR (1))

The stationary first order multiplicative spatial autoregressive model (MSAR (1)) defined by

$$X(s,t) = \alpha X(s-1,t) + \beta X(s,t-1) - \alpha\beta X(s-1,t-1) + \varepsilon(s,t), (s,t) \in \mathbb{Z}^2, \quad (8)$$

where $\varepsilon(s,t)$ are independent identically distributed random variables, this model is stationary if $|\alpha| < 1$ and $|\beta| < 1$ (It can be shown that the MA representation (MSAR (1)) model is

$$b_{i,j} = \begin{cases} \alpha^i\beta^j & , \text{if } i \geq 0, j \geq 0 \\ 0 & , \text{if } i < 0 \text{ or } j < 0. \end{cases}$$

Let $a_{1,0} = \alpha$, $a_{0,1} = \beta$ and $a_{1,1} = -\alpha\beta$, the coefficients $a_{k,l}^{(h_1,h_2)}$, $(h_1,h_2) \geq (0,0)$ and $(k,l) \in \xi = (1,0),(0,1),(1,1)$ are calculated by using the recursive relation (7) and are given by (9) for $h_1 \geq 1, h_2 \geq 1$.

$$a_{k,l}^{(h_1,h_2)} = \begin{cases} a_{k,l+1}^{h_1,h_2-1} + \alpha^{h_1+k}\beta^{h_2+l} & , (k,l) = (1,0), \\ a_{k,l+1}^{h_1,h_2-1} & , (k,l) = (0,1), (9) \\ a_{k,l+1}^{h_1,h_2-1} - \alpha^{h_1+k}\beta^{h_2+l} & , (k,l) = (1,1). \end{cases}$$

The simulation of the MSAR(1) model and the recursive computation of these coefficients were implemented in MATLAB. To evaluate the prediction accuracy, the root mean squared error (RMSE) between the simulated and predicted fields was computed. The RMSE obtained is 1.0503. This value is acceptable as it reflects a relatively low prediction error compared to the scale of the simulated data. Given the inherent randomness introduced by the noise term $\varepsilon(s,t)$ in the MSAR(1) model, an RMSE close to 1 aligns with the standard deviation of the noise ($\sigma = 1$). This indicates that the model effectively captures the spatial dependencies in the random field while accounting for the stochastic variation. Figure 2 shows the simulated random field and the comparison between true and predicted values at one step ahead $(h_1 = 1, h_2 = 1)$.

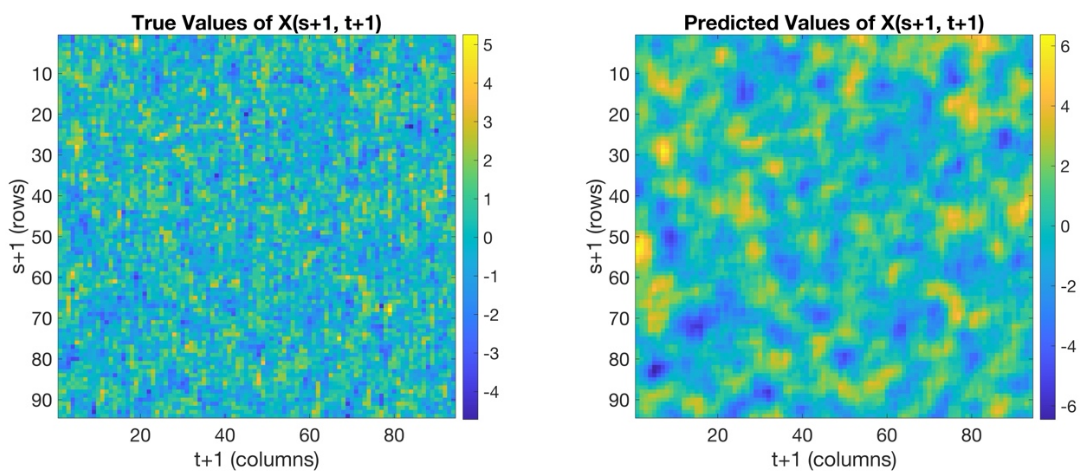

Figure 2. Actual and predicted values of the MSAR (1) model

3.2 Lattice Data Model of Poverty Level in the US State of Texas

As an illustration of a lattice dataset, we analyze the relationship between the poverty level (*pov*) and the total population (*pop*) at the county level in 2009 for the US state of Texas. The data were sourced from the US Census Bureau and enriched with additional geographical information, including latitude and longitude, derived from the Texas shapefile. The full dataset, along with the shapefile, is publicly available in the GitHub repository. The 2D-SAR model derived in this analysis is presented in [9].

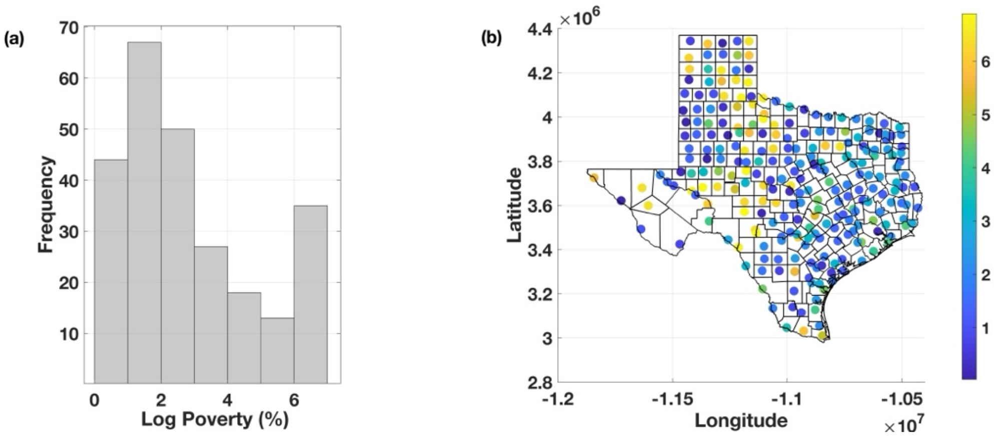

Figure 3. (a) Histogram showing the distribution of log poverty levels (% of the population below 150% of the poverty threshold) across counties in Texas in 2009. (b) Choropleth map of county-level log poverty levels in the state of Texas, with a color scale in

Figure 3 illustrates Texas's 254 counties, color-coded by their poverty levels in 2009. A preliminary analysis shows a strong linear relationship between the logarithm of the poverty level (\log(pov)) and the logarithm of the total population $(\log(pop))$. The least squares regression is given by: $\mathcal{E}\{\log(\text{pop}) \mid pop\} = -1.741 + 0.992 \log(pop)$. Residual analysis from this fit indicates significant spatial association. Specifically, Moran's $I = 0.391$ and Geary's $c = 0.568$ statistics reject the null hypothesis of no spatial association (p-values $< 10^{-15}$). This result highlights substantial spatial clustering among counties' poverty levels, even after adjusting for population size. The SAR model uses the log poverty level as the response variable and the log total population as the explanatory variable, with the neighborhood system based on geographic adjacency: two counties are considered neighbors if and only if their boundaries intersect. The spatial weights are assumed to be $w_{ij} = 1$ for any two neighboring counties s_i and s_j.

The SAR model is fitted using maximum likelihood, resulting in the estimates: $\mathcal{E}\{\log(pov) \mid pop\} = -2.123 + 1.034 \log(pop)$, and $(\hat{\sigma}^2, \hat{\rho}) = (0.067, 0.116)$. The modified AR(1,1) model, where both the intercept and the explanatory variable $x(s,t)$ are included in the error term, is given by:

$$X(s,t) = 0.116 \cdot X(s-1,t) + 0.067 \cdot X(s,t-1) + \epsilon(s,t), (10)$$

where $\epsilon(s,t) = -2.123 + 1.034\ v(s,t) + u(s,t)$, with $u(s,t) \sim \mathcal{N}(0, 0.067)$ being an independent white noise process; $v(s,t)$ representing the logarithm of the population, log(pop), in this region at location (s,t). This model is stationary, if $|a_{1,0}| + |a_{0,1}| < 1$ ([11]). Let $a_{1,0} = \alpha$, $a_{0,1} = \beta$, the MA representation is

$$b_{i,j} = \begin{cases} \alpha^i \beta^j & \text{if } i = 0 \text{ or } j = 0, \\ (i+j)\alpha^i \beta^j & \text{if } i \neq 0, j \neq 0. \end{cases}$$

The coefficients $a_{k,l}^{(h_1,h_2)}$, $(h_1,h_2) \geq (0,0)$ and $(k,l) \in \rho = (1,0),(0,1)$ are calculated by using the recursive relation [7] and are given by

1. For $(h_1,h_2) = (0,0)$, $a_{k,l}^{(0,0)} = a_{k,l}$, $(k,l) \in \rho$,

2. For $h_1 = 0$, $h_2 \neq 0$, $a_{k,l}^{(0,h_2)} = a_{k,l+1}^{(0,h_2-1)} + \alpha^k \beta^{h_2+l}$, $(k,l) \in \rho$,

3. $a_{1,0}^{(0,0)} = \alpha$, $a_{0,1}^{(0,0)} = \beta$.

Now, we apply this relation to estimate the predictor coefficients for various values of h_2. Applying the recursive relation for different values of h_2 provides the following table:

h_2	0	1	2	3
$a_{1,l}^{(0,h_2)}$	0.116	0.232	0.2398	0.2403
$a_{0,l}^{(0,h_2)}$	0.067	0.134	0.1385	0.1388

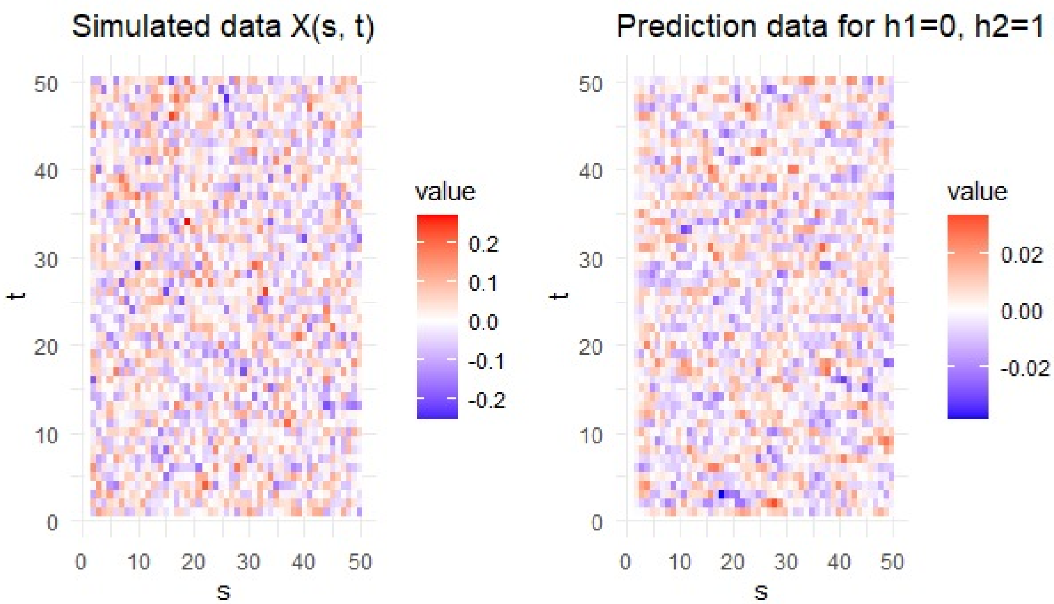

Figure 4. Actual and predicted values of the model

The left panel of Figure 4 shows the simulated 50×50 data grid generated from the poverty data, while the right panel presents the predicted values obtained using the AR(1,1) model with $h_2 = 2$.

4 CONCLUSION

This paper presents a 2D spatial autoregressive model for predicting stationary random fields using the nonsymmetrical half-plane past (NSHP) framework. The methodology is validated through an academic example and a dataset on poverty levels in Texas counties, demonstrating its ability to capture spatial dependencies and improve prediction accuracy. The recursive estimation approach facilitates the systematic computation of predictor coefficients.

REFERENCES

[1] Cressie, N. (2015). Statistics for spatial data. John Wiley & Sons.
[2] Gorsich, D. J., & Genton, M. G. (2000). Variogram model selection via nonparametric derivative estimation. Mathematical geology, 32, 249-270.

[3] Hall, P., & Patil, P. (1994). Properties of nonparametric estimators of autocovariance for sta-tionary random fields. Probability Theory and Related Fields, 99, 399-424.
[4] Im, H. K., Stein, M. L., & Zhu, Z. (2007). Semiparametric estimation of spectral density with irregular observations. Journal of the American Statistical Association, 102(478), 726-735.
[5] Francos, J. M., Meiri, A. Z., & Porat, B. (1993). A unified texture model based on a 2-D Wold-like decomposition. IEEE transactions on signal processing, 41(8), 2665-2678.
[6] Helson, H., & Lowdenslager, D, Prediction theory and Fourier series in several variables, Acta Mathematica, 99(1), 165-202 (1958).
[7] Helson, H., & Lowdenslager, D, Prediction theory and Fourier series in several variables. II, Acta Mathematica, 106(3-4), 175-213 (1961).
[8] Arezki, O., & Hamaz, A. (2022). On linear prediction for stationary random fields with nonsym-metrical half-plane past. Communications in Statistics-Theory and Methods, 51(15), 5298-5309. [9] Oliveira, V., & Trindade, A. A. (2018). Spatial Statistics. In R. Alhajj & J. Rokne (Eds.), Encyclopedia of Social Network Analysis and Mining (pp. 2882–2895). Springer New York.
[10] Kohli, P., & Pourahmadi, M. (2014). Some prediction problems for stationary random fields with quarter-plane past. Journal of Multivariate Analysis, 127, 112-125.
[11] Basu, S., & Reinsel, G. C, Properties of the spatial unilateral first-order ARMA model, Ad- vances in Applied Probability, 25(3), 631-648 (1993).
[12] Bondon, P, Recursive relations for multistep prediction of a stationary time series, Journal of Time Series Analysis, 22(4), 399-410 (2001).
[13] Goryainov, V. B, Least-modules estimates for spatial autoregression coefficients, Journal of Computer and Systems Sciences International, 50(4), 565-572 (2011).
[14] Horv´ath, L., Trapani, L.: Changepoint detection in heteroscedastic random coef- ficient au- toregressive models. Journal of Business & Economic Statistics 41(4), 1300–1314 (2022).
[15] Dunker, F., Mendoza, E., Reale, M.: Regularized maximum likelihood estimation for the random coefficients model. Econometric Reviews 44(2), 192–213 (2024).