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# Bifurcation and Pattern Variation of Mixed Convection in a Vertical Channel

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***Abstract*** -The stability of stably stratified mixed convective flow in a vertical channel is examined by use of weakly nonlinear stability analysis. The nonlinear results are presented for fluid as water (Prandtl number ($Pr$) equal to 7) and a fixed Reynolds number $Re$ equal to 1000. The weakly nonlinear analysis predicts only supercritical bifurcation at and beyond the critical Rayleigh number ($Ra$). The equilibrium amplitude increases beyond the critical point. Due to nonlinear interaction, a substantial enhancement in heat transfer rate is observed from the basic state beyond the bifurcation point. The the impact of nonlinear interaction of waves on pattern of secondary flow is also studied and found it distorts the fundamental wave shape.

***Keywords:*** Finite amplitude, non-isothermal, nonlinear stability, Poiseuille flow

## 1 Introduction

Mixed convective flow through vertical ducts is encountered in a wide class of application such as nuclear reactor, heat exchanger, electronic equipment, etc. The instability and transition characteristics of non-isothermal flow (mixed convective flow) differ substantially from those of an isothermal flow (Scheele and Hanratty, 1962; Yao, 1987). For example, mixed convective pipe flows become unstable even under mild heating conditions at low Reynolds number (Scheele and Hanratty, 1962; Kemeny and Somers, 1962). These instabilities occur mainly by thermal effects.

Linear stability theory addresses the first stage of the transition process, and it is used to find the location of bifurcations as well as predict the form of the developing disturbances. However, it is inadequate to describe the instabilities in some flows. For example, the plane Poiseuille flow becomes linearly unstable at a Reynolds number of 5772, but the transition for this flow appears at much low Reynolds number in reality. Similarly, the isothermal pipe flow undergoes transition at a Reynolds number $O(2000)$, which is linearly stable at all Reynolds numbers.

When stable or unstable disturbances reach appreciable amplitudes, then it becomes difficult to explain the stability of the flow by linear theory. In general, the transition from smooth laminar to disorder turbulent flow can involve a sequence of instabilities in which the system realizes progressively more complicated states (Niemela et al., 2000) or it can occur suddenly (Grossmann, 2000; Hof et al., 2004). In the former case, the complexity arises in well defined steps in the name of bifurcation sequences (super or subcritical). In this situation, the mode that will be amplified can not be clear from the linear stability analysis because any of the potential unstable waves may grow and interact with other modes. The prediction of wideband nature of sequence of instabilities of the detailed flow pattern and temperature distribution at a point away from the critical is beyond the scope of linear stability analysis. In this situation to understand the complexity in the sequence of instabilities of the flow one can take the help of a weakly nonlinear stability analysis.

Recently, Khandelwal and Bera (2015) have investigated the detailed weakly nonlinear stability analysis of mixed convective flow in a vertical channel for fluid as mercury, gases liquids and heavy oils, whose linear stability analysis is already examined by Chen and Chung (1996). Their investigation is mainly concerned with fluids as mercury and gases. They have studied the influence of nonlinear interaction of different harmonics in terms of amplitude function on other physical aspects such equilibrium amplitude, heat transfer rate, friction coefficient, wavespeed, energy spectrum and secondary flow pattern. They have also found the transition phenomena in stably stratified flow is supercritical, which coincides with prediction by direct numerical simulation (DNS) (Chen and Chung, 2003). The detailed instability mechanism of water is not investigated in the above study. Therefore, in the present paper a step has been taken in this direction. The analysis is carried out to determine the nature of the instability, amplitude behavior of unstable disturbance as well as pattern variation. This analysis is centered around the derivation of the Landau equation to calculate the nonlinear interaction of different harmonic waves.

## 2 Mathematical Formulation

A pressure-driven non-isothermal flow in a vertical channel of width $2L$ is shown in Fig 1. The wall temperature of the channel is assumed to vary linearly with $x$ as $T_w = T_0 + Cx$, where $C$ is a positive constant and $T_0$ is the upstream reference temperature. The gravitational force is aligned in the negative $x$-direction. The thermo-physical properties of the fluid are assumed to be constant except for density dependence of the buoyancy term in the momentum equation, which is satisfied by the Boussinesq approximation.

The non-dimensional space coordinates $(x^*, y^*, z^*)$, dependent variables $(V^*, \theta^*, P^*)$ and time $t^*$ are calculated after scaling the dimensional variables as follows:

$$(x^*, y^*, z^*) = \frac{(x,y,z)}{L},\ V^* = \frac{V}{\overline{U}_o},\ \theta = \frac{(T-T_w)}{CLPrRe},\ P^* = \frac{P}{\rho \overline{U}_o{}^2},\ t^* = \frac{t\overline{U}_o}{L}, \tag{1}$$

where, $V^* = (u^*, v^*, w^*)$, $\theta$, $P^*$ and $t^*$ are the dimensionless velocity vector, temperature, pressure and time respectively. Furthermore, $\overline{U}_o$ and $\rho$ are dimensional average base velocity and reference density of the fluid, respectively. The nondimensional governing equations, after dropping asterisks, can be written as (Chen and Chung, 1996)

$$\nabla \cdot V = 0, \tag{2}$$

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}.\nabla\mathbf{V} = -\nabla P + \frac{Ra}{Re}\theta\mathbf{e_x} + \frac{1}{Re}\nabla^2\mathbf{V}, \tag{3}$$

$$\frac{\partial \theta}{\partial t} + \mathbf{V}.\nabla\theta = \frac{1}{RePr}(\nabla^2\theta - u). \tag{4}$$

The nondimensional parameters appearing in the problem are the Rayleigh number $(Ra)$, Reynolds number $(Re)$ and Prandtl number $(Pr)$. They are defined as $Ra = \frac{g\beta_T CL^4}{\widetilde{\nu}k}$, $Pr = \frac{\widetilde{\nu}}{k}$ and $Re = \frac{\overline{U}_o L}{\widetilde{\nu}}$, where $k$, $\widetilde{\nu}$, $\beta_T$ and g are the thermal diffusivity, kinematic viscosity, thermal expansion coefficient and gravitational acceleration, respectively. The notation $\mathbf{e_x}$ in the equation (3) denotes unit vector in the $x$-direction.

### 2.1 Linear Stability Analysis

The basic flow, whose stability is to be analyzed, is steady, unidirectional and fully developed. Under these assumptions the governing Eqs. (2) -(4) are reduced into the following set of equations:

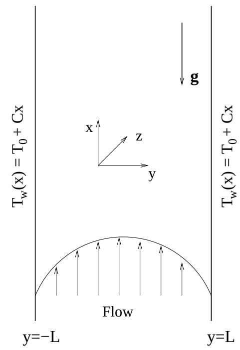


Fig. 1: Schematic of the problem considered.

$$\frac{d^2U_o}{dy^2} + Ra\Theta_o = Re\frac{dP_o}{dx}, \tag{5}$$

$$\frac{d^2\Theta_o}{dy^2} - U_o = 0. \tag{6}$$

The solution of the basic flow Eqs. (5)-(6) admits the boundary conditions: $U_o = \Theta_o = 0 \quad \text{at} \quad y = \pm 1$, where $U_o$, $\Theta_o$ and $P_o$ are the basic state velocity, basic state fluid temperature and basic state pressure, respectively.

To examine the linear stability of the above basic state, the dependent variables are split into normal mode form (Drazin and Reid, 2004) as

$$V(x,y,z,t) = U_o(y)\mathbf{e_x} + \hat{V}(y)e^{i(\alpha x+\beta z-\alpha ct)}, \tag{7a}$$

$$\theta(x,y,z,t) = \Theta_o(y) + \hat{\theta}(y)e^{i(\alpha x+\beta z-\alpha ct)}, \tag{7b}$$

$$p(x,y,z,t) = P_o(x) + \hat{p}(y)e^{i(\alpha x+\beta z-\alpha ct)}, \tag{7c}$$

where hat denotes a small disturbance quantity, $\hat{V} = (\hat{u},\hat{v},\hat{w})$, $\alpha$ and $\beta$ are the wavenumber in the $x$ and $z$ directions respectively, and $c = c_r + ic_i$ is the complex wavespeed. The linearized disturbance equations can be seen in the paper of Chen and Chung (1996).

Linear stability equations form a generalized eigenvalue problem for a complex disturbance wavespeed ($c$). The disturbance is stable or neutrally stable or unstable as $c_i$ is negative or zero or positive, respectively. The detailed analysis of linear stability characteristics of the above flow is examined by Chen and Chung (1996). The objective of present study is to analyze the nonlinear stability analysis of the non-isothermal Poiseuille flow, when fluid is water.

The overview of linear stability results provides some information regarding the development of the disturbance. However, it can not provide any information about the amplitude of such disturbances and quantitative information about a disturbed flow. Therefore, a nonlinear stability is required to study the structure of the flow field that results from linear stability.

## 3 Weakly Nonlinear Analysis

The linear theory of hydrodynamic stability predicts only the onset of instability to infinitesimal amplitude. When the larger amplitude are obtained the linear theory of stability is questionable. It does not provide quantitative information about the actual size of the disturbances. The size of the disturbances may be finite or not. The nonlinear effects modify the growth rate of disturbance predicted by linear theory. Therefore, nonlinear stability analysis is necessary to find the quantitative information about a disturbed flow, such as amplitudes of the disturbances, growth and decay of the disturbances

To study the finite amplitude instability using weakly nonlinear theory, the dependent variables are first separated into Fourier components of a disturbance wave predicated by linear instability theory. The equations governing the harmonic components are then solved using a perturbation expansion. The Fourier expansion of the $x$-direction velocity in separable form is:

$$u(x,y,z,t) = U(y,\tau)E^0 + \hat{u}_1(y,\tau)E^1 + \hat{u}_2(y,\tau)E^2 + ... + c.c. \tag{8}$$

where, $E = e^{[i\alpha(x-c_r t)+i\beta z]}$, $\alpha$ is the wavenumber corresponding to the critical $Ra$ and $c_r$ is the real part of wavespeed of the most unstable disturbance wave. c.c. stands for complex conjugate. The inclusion of higher order harmonics is not necessary to develop a cubic Landau equation.

The function for the harmonic components are further decomposed by expanding in terms of the small parameter. Using the method of multiple timescales, the slow timescale $\tau = c_i t$ is given as

$$\frac{\partial}{\partial t} \equiv \frac{\partial}{\partial t} + c_i \frac{\partial}{\partial \tau}. \tag{9}$$

The following expansion of the $E^1$ wave is consistent

$$\hat{u}_1(y,\tau) = (c_i)^{\frac{1}{2}} B u_{10} + (c_i)^{\frac{3}{2}} B|B|^2 u_{11} + O((c_i)^{\frac{5}{2}}) \tag{10}$$

where, $B$ denotes the amplitude function of order one, which will be calculated from Landau equation. c.c. stands for complex conjugate. The expansion of $E^1$ wave leads to the following forms for $E^0$ and $E^2$ waves

$$U(y,\tau) = U_o(y) + c_i|B(\tau)|^2 U_1(y) + O((c_i)^2) \tag{11}$$

$$\hat{u}_2(y,\tau) = c_i B^2 u_{20} + O((c_i)^2) \tag{12}$$

The system of harmonic equations are obtained by substituting Eqs. (8)-(12) into the governing equations Eqs. (2)-(4). The different order harmonic equations can be solved sequentially in increasing power of $c_i$. The detail of system of harmonic equations and its solution procedure is given in the reference Khandelwal and Bera (2015).

At order $(c_i)^{3/2}$, the equations of harmonic $E^1$ become non-homogeneous equations. The left hand side of these equations contains linear stability operators operating on $u_{11}$, $v_{11}$, $w_{11}$ and $\theta_{11}$, and right hand side of these equations contains the terms proportional to $dB/d\tau$, $B$ and $B|B|^2$. The coefficients of the terms on the right-hand sides are known from the lower-order analysis. Since the homogeneous forms of the equations of $E^1$ are exactly same as linear stability theory, the integrability condition requires that the right-hand side terms must be orthogonal to the functions satisfying the homogeneous adjoint problem. This leads to the following cubic Landau equation for the disturbance amplitude function $B$:

$$\frac{dB}{d\tau} = \alpha B + a_1 B|B|^2, \tag{13}$$

The constant $a_1$ is known as the first Landau constant, and it is obtained through the integrability condition, which is given in Khandelwal and Bera (2015). The subcritical or supercritical bifurcation (instability ) of the flow depends on the sign of the real part of $a_1$. If the real part of $a_1$ is positive (negative) then we predict a subcritical (supercritical) type of bifurcation. The equilibrium amplitude (threshold amplitude) of supercritical (subcritical) is ${A_e}^2 = -\alpha c_i/(a_1)_r ({A_t}^2 = |\alpha c_i/(a_1)_r|)$, where $(a_1)_r$ is the real part of the Landau constant. The actual value of the Landau constant depends on the chosen normalization of eigenvectors obtained from linear stability.

## 4 Results and Discussion

The finite amplitude instability of stably stratified non-isothermal Poiseuille flow in a vertical channel is studied by means of weakly nonlinear stability theory. The governing parameters are Reynolds number ($Re$), Prandtl number ($Pr$) and Rayleigh number ($Ra$). The influence of nonlinear interaction of different harmonic modes on the flow instability is examined for fluid as water ($Pr = 7$). The Reynolds number is fixed at 1000 for this study. Linear stability analysis (Chen and Chung, 1996) shows that the minimum critical point is obtained for $\beta = 0$. Therefore, the present nonlinear stability analysis is also spanwise independent. The critical Rayleigh number ($Ra_c$) and wavenumber ($\alpha_c$) for above set of parameter are 15.61 and 0.024 respectively. The numerical validation of the present study can be seen in the article (Khandelwal and Bera, 2015).

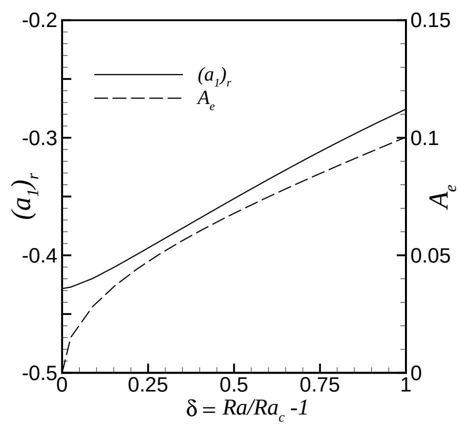


Fig. 2: Variation of real part of Landau constant $(a_1)_r$ and equilibrium amplitude ($A_e$)

The variation of real part of the Landau constant and equilibrium amplitude is plotted in Fig. 2. The abscissa of the figure is normalized by $\delta = Ra/Ra_c - 1$. As can be seen from above figure the value of $(a_1)_r$ is negative at and beyond the critical $Ra$, which leads a supercritical bifurcation corresponding to the most unstable linear wave for linearly unstable flow. No regions of subcritical type of instability are obtained for larger values of Rayleigh number in stably stratified flow with respect to most linearly unstable wave. Chen and Chung (2003) have also mentioned that the flow transition phenomena of non-isothermal Poiseuille flow in a vertical channel is always supercritical by direct numerical simulation, which is the excellent support of the present study. Similar result for stably stratified Poiseuille flow is also obtained in the pipe (Scheele and Hanratty, 1962) and annulus (Yao and Rogers, 1992). The equilibrium amplitude ($A_e$) for supercritical bifurcation away from the critical point is also plotted in Fig. 2. The equilibrium amplitude increases smoothly beyond the critical $Ra$ (taken upto two times of critical). The increase in the equilibrium amplitude of most unstable disturbance wave can be explained by the definition of $A_e$. A smooth increase in $A_e$ appears due to combined impact of $\alpha c_i$ and $|(a_1)_r|$ for the entire range of Rayleigh number.

We have discussed about the type of bifurcation and variation of equilibrium amplitude of the most unstable disturbance beyond the critical point. Now we are curious to know the impact of nonlinear interaction of different harmonics in terms of amplitude function on heat transfer rate as well as secondary flow pattern.

The evaluation of equilibrium disturbance amplitude is necessary to predict the heat transfer rate for disturbed flow. The characteristics of the disturbed flow on Nusselt number ($Nu$) is examined in Fig. 3. The average rate of heat transfer is given as

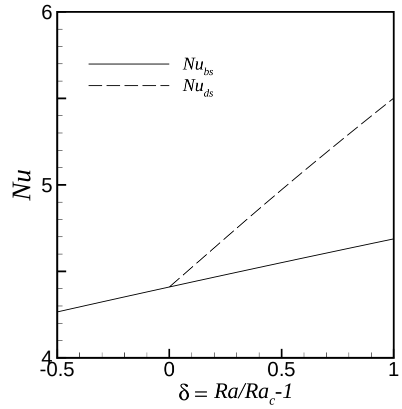


Fig. 3: Variation of Nusselt number as function of Rayleigh number. $Nu_{bs}$: Nusselt number of basic state, $Nu_{ds}$: Nusselt numbers of distorted flow by nonlinear analysis

$$Nu = -2\frac{\partial \Theta}{\partial y}|_{y=1} \Bigg/ \frac{\int_{-1}^{1} U\Theta dy}{\int_{-1}^{1} U dy}. \tag{14}$$

In Fig. 3 $Nu_{ds}$ is calculated with the help of equilibrium amplitude. The functions $U$ and $\Theta$ for $Nu_{ds}$ are given by $U_o + A_e^2 U_1$ and $\Theta_o + A_e^2 \Theta_1$, respectively. The results of this calculation for $Ra > 0$ at critical wavenumber ($\alpha$) are given by the single wave result. The results show that the Nusselt number for distorted flow is much more compared to the Nusselt number predicted by the basic state. Quantitatively, we have found that the increase in $Nu$ due to nonlinear interaction is 10% at $\delta = 0.5$ and 18% at $\delta = 1$. This indicates that the nonlinear interaction of different modes causes a substantial increase of Nusselt number. This demonstrates that heat transfer correlation obtained analytically by use of parallel flow assumption is inadequate in mixed convection. The increase in heat transfer rate due to instability was also observed for $Pr = 6$ experimentally by Maitra and Subba Raju (1975), and theoretically by Yao and Rogers (1992) while studying non-isothermal flow in a vertical annulus. We know that in this flow a large waveband will become unstable at nearly the same time. Therefore, as $Ra$ increases all of these unstable waves must be included in the model to unsteady flow pattern. Consequently, it will induce transverse mixing of the fluid, and enhance the heat transfer rate.

To shed more light on the mechanism of supercritical/subcritical bifurcation, an investigation of the energy transfer due to change of shape of the fundamental disturbance wave on the secondary flow pattern is made. We have studied the pattern of secondary flow within one period under linear as well as nonlinear stability theories.

Figure 4 is plotted to analyze the supercritical bifurcation in terms of pattern variation of the secondary flow of disturbed velocity and temperature component within one period for $\delta = 0.5$ (or $Ra = 23.41$). These figures are plotted at the corresponding most unstable linear wave. The solid and dashed lines in contours are associated with respective positive and negative values of the field variable, and defined as even and odd cells, respectively. The contour plots for disturbance velocity components and disturbance temperature $((i)-(iii))$ by linear stability analysis at $\delta = 0.5$ are shown in Fig. 4(a), whereas Fig. 4(b) shows the same by nonlinear stability analysis. The supercritical bifurcation due to the interaction of different harmonic modes stretches the cells of velocity and temperature disturbance components towards the walls of the channel. So, the nonlinear interaction distorts the fundamental wave. Apart from this the clockwise and anti-clockwise cells are shifted from their respective places in the same period due to the interaction of superimposed

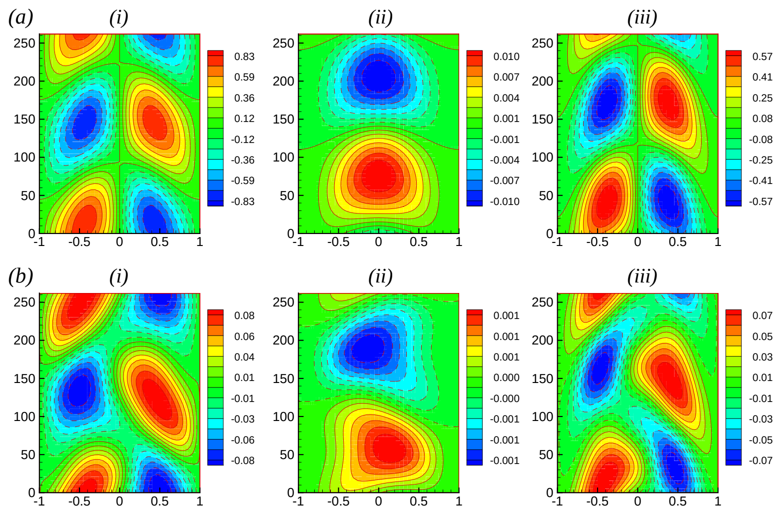


Fig. 4: Pattern of secondary flow by linear stability (*a*) and by nonlinear stability (*b*) at $\delta = 0.5$: (*i*) disturbed $u$-velocity, (*ii*) disturbed $v$-velocity and (*iii*) disturbed temperature.

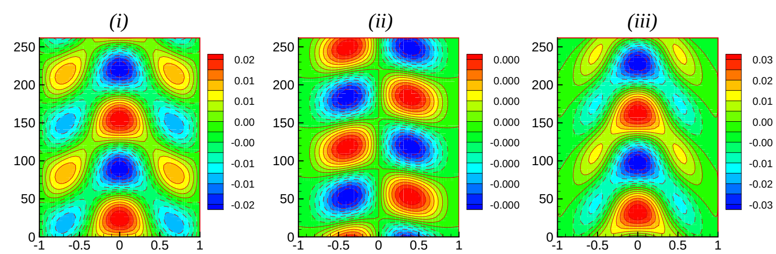


Fig. 5: Pattern of $E^2$ component in the secondary flow at $\delta = 0.5$; (*i*) disturbed $u$-velocity, (*ii*) disturbed $v$-velocity and (*iii*) disturbed temperature.

harmonic modes. Since our analysis is restricted upto $E^2$ harmonic mode and the pattern of secondary flow due to fundamental mode $(E^1)$ is similar to the pattern of secondary flow by linear stability. Therefore, the stretching of cells appears due to interaction of $E^2$ harmonic mode. Figure 5 is plotted for $E^2$ harmonic mode at $\delta = 0.5$. As can be seen from the above figures, compared to secondary flow due to $(E^1)$ the same for $E^2$ contains many small even and odd cells. The magnitude of these cells are very small compared to the same for $E_1$. The number of cells and magnitude of $u$-velocity component is much more than the same for $v$-component. Since buoyancy acts in the direction of $u$-component velocity only, therefore modification in the buoyant production acts as a key role on the pattern of the $u$-component velocity as well as temperature.

## 5 Conclusions

In this work, a weakly nonlinear stability of non-isothermal Poiseuille flow in a vertical channel is examined, whose linear stability is performed by Chen and Chung (1996). The results are presented for fluid as water and $Re = 1000$. The calculation of the first Landau constant shows supercritical bifurcation at and beyond the bifurcation point (critical point), in agreement with direct numerical simulation result (Chen and Chung, 2003). The equilibrium amplitude in this flow increases beyond the bifurcation point. The impact of nonlinear interaction on heat transfer rate significant, i.e. Nusselt number predicted by nonlinear analysis is much more than those predicted by fully developed basic state. The nonlinear impact is also examined in terms of pattern of secondary flow. The interaction of different harmonic modes stretches the cells of velocity and temperature disturbance components, i.e. it distorts the fundamental wave.

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