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Unsteady Stagnation-Point Flow of a Second-Grade Fluid

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Abstract -The unsteady two-dimensional stagnation point flow of second-grade fluid impinging on an infinite plate is examined and solutions are obtained. It is assumed that the infinite plate at $y = 0$ is making harmonic oscillations in its own plane. Solutions for small and large frequencies of the oscillations are obtained for various values of the Weissenberg number. The effect of the Weissenberg number is to decrease the velocity near the wall as it increases.

***Keywords**:* unsteady, stagnation-point, oscillating plate, non-Newtonian fluid.

1. Introduction

In the past, the fluid flow near a stagnation-point has been investigated extensively. In his pioneer work, Hiemenz (1911) derived an exact solution of the steady flow of a Newtonian fluid impinging orthogonally on an infinite flat plate. The stagnation-point flow when the fluid impinges obliquely on the plate has been studied independently by Stuart (1959), Tamada (1979) and Dorrepaal (1986). Beard and Walters (1964) used boundary-layer equations to study two-dimensional flow near a stagnation point of a non-Newtonian viscoelastic fluid. The behaviour of a viscoelastic fluid impinging on a flat rigid wall at an arbitrary angle of incidence was analyzed by Dorrepaal et al (1992).

The unsteady stagnation-point flow of a Newtonian fluid has also been studied extensively. Rott (1956) and Glauert (1956) studied the stagnation point flow of a Newtonian fluid when the plate performs harmonic oscillations in its own plane. Matunobu (1977) examined the fundamental character of the unsteady flow near a stagnation point for a Newtonian fluid. Takemitsu and Matunobu (1979) studied the oblique stagnation point flow for a Newtonian fluid and obtained the general features of a periodic stagnation point flow. The case when the stagnation point fluctuates along a solid boundary is especially interesting from the biomechanical point of view. This is because the wall shear stress experienced by blood vessels may be thought to be increased by pulsating blood flow near the mean position of fluctuating stagnation point and lead to vascular diseases.

In this work, the unsteady stagnation point flow of a viscoelastic second-grade fluid is examined and solutions are obtained. We assume that the infinite plate at $y = 0$ is oscillating with velocity $U \cos \Omega t$, the fluid occupies the entire upper half plane $y > 0$ and the fluid impinges obliquely on the plate. The governing partial differential equations are reduced to a system of ordinary differential equations by assuming a form of the streamfunction a priori. The resulting equations are, then, solved numerically using a shooting method for various values of the Weissenberg number, We. It is observed that the effect of the Weissenberg number is to decrease the velocity near the wall as it increases. Furthermore, analytical solutions are obtained for small and large values of frequency.

2. Flow Equations and Solutions

The two-dimensional flow of a viscous incompressible non-Newtonian second-grade fluid, neglecting thermal effects and body forces, is governed by

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right)-\frac{\alpha_1}{\rho}\frac{\partial}{\partial t}\left(\nabla^4\psi\right)-\frac{\partial\left(\psi,\nabla^2\psi\right)}{\partial\left(x,y\right)}+\frac{\alpha_1}{\rho}\frac{\partial\left(\psi,\nabla^4\psi\right)}{\partial\left(x,y\right)}-\nu\nabla^4\psi=0 \tag{1}$$

where $\psi=\psi(x,y,t)$ is the streamfunction , $\nu=\dfrac{\mu}{\rho}$ is the kinematic viscosity, α_1, α_2 are the normal stress moduli. Having obtained a solution of equation (1), the velocity components are given by

$$u=\frac{\partial\psi}{\partial y},\quad v=-\frac{\partial\psi}{\partial x} \tag{2}$$

The shear stress component τ_{12} is given by

$$\tau_{12}=\mu\left[\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right]+\alpha_1\left[\frac{\partial\psi}{\partial y}\left(\frac{\partial^3\psi}{\partial x\partial y^2}-\frac{\partial^3\psi}{\partial x^3}\right)-\frac{\partial\psi}{\partial x}\left(\frac{\partial^3\psi}{\partial y^3}-\frac{\partial^3\psi}{\partial x^2\partial y}\right)\right.$$
$$\left.+2\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^2\psi}{\partial y^2}+2\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial x\partial y}\right] \tag{3}$$

Following Takemitsu and Matunobu (1979), we assume that

$$\psi=k\left[x\,f(y)+g(y,t)\right] \tag{4}$$

We assume that the infinite plate at $y=0$ is oscillating with velocity $Ucos\,\Omega t$ and that the fluid occupies the entire upper half plane $y>0$. Furthermore, we assume the streamfunction far from the wall is given by $\psi=\frac{1}{2}\gamma y^2+xy$. Thus, the boundary conditions are given by

$$f(0)=f'(0)=0,\qquad g(0,t)=0,\qquad g_y(0,t)=\frac{U}{k}e^{i\Omega t} \tag{5}$$

$$f'(\infty)=1,\qquad g_y(\infty,t)=\gamma y \tag{6}$$

where γ is a non-dimensional constant characterizing the obliqueness of oncoming flow. It is assumed that only the real part of a complex quantity has its physical meaning.

Substituting equation (4) in (1) and upon integration with respect to y yields after employing the conditions at infinity.

$$\nu f'''+k\left(f\,f''-f'^2\right)-\frac{\alpha_1 k}{\rho}\left(f\,f^{(iv)}-2f'\,f'''+f''^2\right)=-k \tag{7}$$

And

$$\nu\frac{\partial^3 g}{\partial y^3}-\frac{\partial^2 g}{\partial t\partial y}+\frac{\alpha_1}{\rho}\frac{\partial^4 g}{\partial t\partial y^3}+k\left(f\frac{\partial^2 g}{\partial y^2}-f'\frac{\partial g}{\partial y}\right)-\frac{\alpha_{1k}}{\rho}\left(f\frac{\partial^4 g}{\partial y^4}-f'\frac{\partial^3 g}{\partial y^3}+f''\frac{\partial^2 g}{\partial y^2}-f'''\frac{\partial g}{\partial y}\right)=0 \tag{8}$$

Using the non-dimensional variables

$$\eta = \sqrt{\frac{k}{\nu}}\, y, \quad \tau = \Omega t, \quad f(y) = \sqrt{\frac{\nu}{k}}\, F(\eta), \quad g(y,t) = \frac{\nu}{k} G(\eta, \tau), \quad \varepsilon = \frac{U}{\sqrt{\nu k}} \tag{9}$$

in equations (7) and (8), and boundary conditions (5) and (6), we obtain

$$\begin{gathered} F''' + F F'' - F'^2 - W_e\left(F F^{(iv)} - 2F' F''' + F''^2\right) = -1 \\ F(0) = 0, \qquad F'(0) = 0, \qquad F'(\infty) = 1 \end{gathered} \tag{10}$$

and

$$\begin{gathered} \frac{\partial^3 G}{\partial \eta^3} + F\frac{\partial^2 G}{\partial \eta^2} - F'\frac{\partial G}{\partial \eta} - We\left(F\frac{\partial^4 G}{\partial \eta^4} - F'\frac{\partial^3 G}{\partial \eta^3} + F''\frac{\partial^2 G}{\partial \eta^2} - F'''\frac{\partial G}{\partial \eta}\right) - \beta\frac{\partial^2 G}{\partial \tau \partial \eta} + We\beta\frac{\partial^4 G}{\partial \tau \partial \eta^3} = 0 \\ G(0,\tau) = 0, \quad G_\eta(0,\tau) = \varepsilon e^{i\tau}, \quad G_{\eta\eta}(\infty,\tau) = \gamma \end{gathered} \tag{11}$$

where We is the Weissenberg number.

Using the quasi-linearization technique described by Garg and Rajagopal (1990), we find that $F''(0) = 1.23259$ when $We = 0$ which is in good agreement with the value obtained by Takemitsu and Matunobu (1979). Numerical values of $F''(0)$ for different values of We are shown in Table 1. These values are in good agreement with the values obtained by Garg and Rajagopal (1990). Figure 1 shows the profiles of F' for various values of We. We observed that as the elasticity of the fluid increases, the velocity near the wall decreases.

Letting $G(\eta,\tau) = G_0(\eta) + \epsilon G_1(\eta)e^{i\tau}$ and $G_0'(\eta) = \gamma H_0(\eta)$, then system (11) gives

$$H_0'' + FH_0' - F'H_0 - We(FH_0''' - F'H_0'' + F''H_0' - F'''H_0) = 0 \tag{12}$$

$$H_0(0) = 0, \quad H_0'(\infty) = 1 \tag{13}$$

and

$$G_1''' + FG_1'' - F'G'_1 - We\left(FG_1^{(iv)} - F'G_1''' + F''G_1'' - F'''G_1'\right) - i\beta(G_1' - We\, G_1''') = 0 \tag{14}$$

$$G_1(0) = 0, \quad G_1'(0) = 1, \quad G_1'(\infty) = 0 \tag{15}$$

System (12)-(13) is solved numerically using a shooting method and it is found that for $We = 0$, $H_0'(0) = 0.607965$. Since $G_0''(0) = \gamma H_0'(0)$, then for $We = 0$, $G_0''(0) = 0.607965\,\gamma$ which is in good agreement with the value obtained by Takemitsu and Matunobu (1979). Numerical values of $H_0'(0)$ for different values of We are shown in Table 1. Figure 2 depicts the profiles of H_0' for various values of We.

Letting $\phi(\eta) = G_1'(\eta)$, then system (14)-(15) becomes

$$\phi'' + F\phi' - F'\phi - We(F\phi''' - F'\phi'' + F''\phi' - F'''\phi) - i\beta(\phi - We\,\phi'') = 0 \tag{16}$$

$$\phi(0) = 1, \quad \phi(\infty) = 0 \tag{17}$$

For small values of the frequency β, we assume that

$$\phi(\eta) = \phi_0\,(\eta) + i\beta\,\phi_1(\eta) + (i\beta)^2\phi^2(\eta) + \cdots \tag{18}$$

where the numerical values for $\phi_0'(0)$, $\phi_1'(0)$ and $\phi_2'(0)$ are given in Table 1 for different values of We.

For large values of the frequency β, we let $Y = \alpha\eta$, $\alpha = \sqrt{i\beta}$ and we found that

$$\phi'(0) \;=\; \phi_0{}'(0) + \alpha\phi_1{}'(0) + \alpha^2\phi_2{}'(0) + \alpha^3\phi_3{}'(0) + \ldots \tag{19}$$

$$= -\frac{1}{\sqrt{1+m}} - \frac{(3-4m)}{8(1+m)}F''(0)\,\alpha^3 + \frac{3+4m}{16\sqrt{1+m}}\alpha^4 \tag{20}$$

$$-\frac{(40m^3-50m^2+28m-33)F''^2(0)}{128(1+m)\sqrt{1+m}}\,\alpha^6 + \cdots \tag{21}$$

provided $m \neq 1$.

3. Conclusions

The unsteady second grade stagnation-point flow impinging obliquely on an oscillatory flat plate is studied. Numerical results for this flow are found for various values of the Weissenberg number We. Figure 1 shows the variations of $F'(\eta)$ for various values of We. The effect of the Weissenberg number, We, is to decrease the velocity $F'(\eta)$ near the wall as it increases. Figure 2 depicts the variations of $H_0'(\eta)$ for various values of We and shows that $H_0'(\eta)$ decreases near the wall as We is increasing.

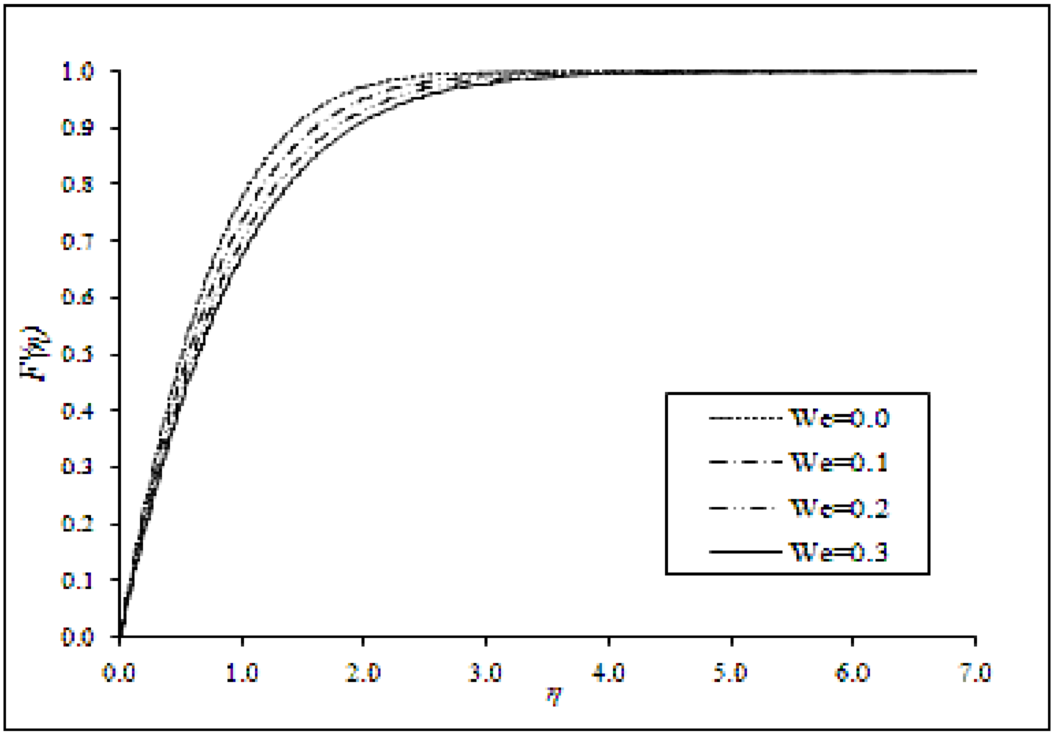

Fig. 1. Variations of $F'(\eta)$ for various values of We.

Table. 1. Numerical values of $F''(0)$, $H_0'(0)$, $\phi_0'(0)$, $\phi_1'(0)$ and $\phi_2'(0)$ for different values of We.

We	$F''(0)$	$H_0'(0)$	$\phi_0'(0)$	$\phi_1'(0)$	$\phi_2'(0)$
0.0	1.23259	0.60777	-0.81107	-0.49348	0.09471
0.5	0.90248	0.39774	-0.65619	0.51922	-0.05474
1.0	0.75276	0.30691	-0.57522	0.51155	-0.15428
2.0	0.59677	0.21662	-0.48170	0.48461	-0.27270
10	0.30283	0.07127	-0.27371	0.34885	-0.38807

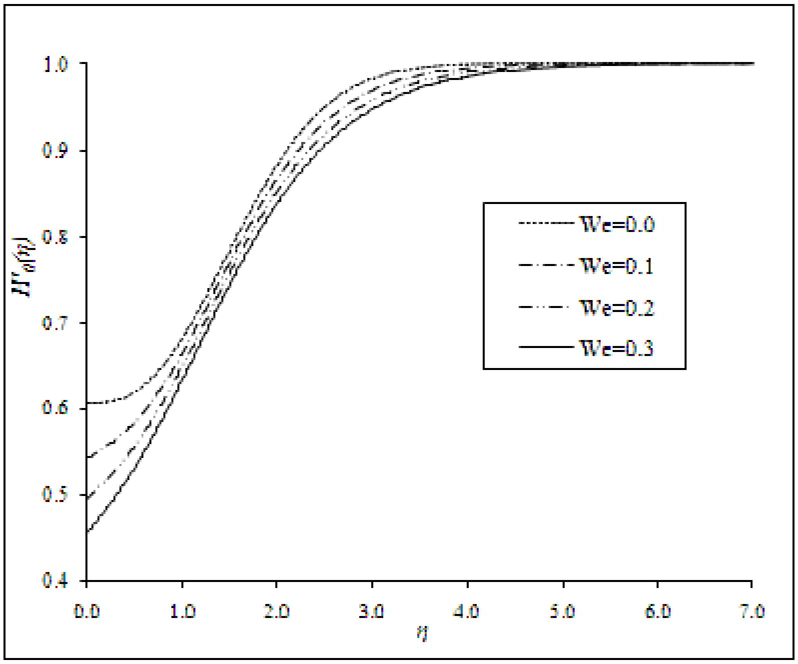

Fig. 2. Variations of $H_0'(\eta)$ for various values of We.

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