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# A Re-examination of Power Coefficient as a Measure of Performance for Horizontal Axis Wind Turbines

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***Abstract* -** Linear momentum theory as applied to horizontal axis wind turbines (HAWTs) provides perhaps the most useful basis for understanding their operation. In particular, the theoretically derived expression for power coefficient represents a convenient measure of performance, as well as provides insight into optimal operating conditions. The typical interpretation of power factor as an energy conversion efficiency, however, especially in the context of converting the "power in the wind" to a power output, has several conceptual difficulties. In this paper it is argued that the energy efficiency interpretation of power coefficient can be misleading, potentially leading to misinterpretation of performance of different wind turbine designs. Instead, an interpretation of power coefficient as the "relative capture area" of a wind turbine is suggested, analogous to the relative capture width parameter for ocean wave energy conversion devices. Such an interpretation gives a more physically coherent picture of wind turbine performance and provides a more pragmatic measure of performance, one that can also be applied to other wind machine designs.

***Keywords*:** Power coefficient, Betz limit, capture width

## 1. Introduction: Linear Momentum Theory, Power Coefficient, and the Betz Limit

Linear momentum theory applied to the operation of horizontal axis wind turbines (HAWTs) dates back at least to Betz [1] and provides one of the most accessible bases for conceptualizing the function of such devices. Given that many thorough treatments of the topic exist in the literature [2]-[4] we give only the most salient features here.

Figure 1 gives a side view of the analysed system, which consists of a diverging stream tube extending from the downstream side of the wind turbine to the upstream side. The variables $V$ and $A$ refer to wind speed and cross-sectional area, respectively, whereas $F_T$ is the thrust force exerted on the hub by the wind and $\dot{W}_{out}$ is the power extracted by the turbine. The subscripts indicate the planes in which the windspeed corresponds to its undisturbed, freestream value ($i$), the value at the hub of the wind turbine ($t$), and the minimum downstream value before the wind reforms ($e$). Local static pressure varies in the flow direction but takes on atmospheric values at locations ($i$) and ($e$). The hub is considered an "actuator disc" where pressure changes discontinuously and across which all energy extraction occurs. The flow is modelled as incompressible with density $\rho$, one-dimensional, isothermal, and frictionless.

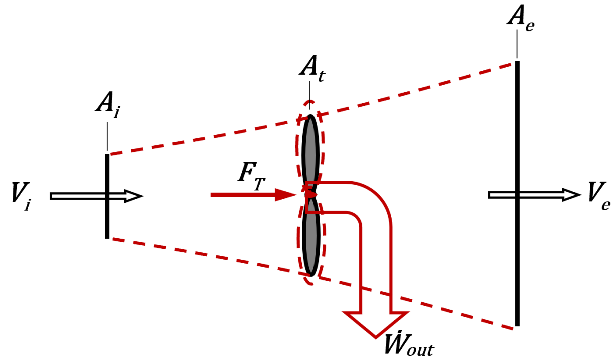


Fig. 1: Diagram of stream tube used in the application of linear momentum theory.

Macroscopic mass, momentum, and energy balances applied to the system in Fig. 1 result in a predicted value for developed power given by

$$\dot{W}_{out} = 4a(1-a)^2\ \frac{1}{2}\rho V_i^3 A_t, \tag{1}$$

where $a$ is the axial induction factor, a parameter representing the fractional decrease in windspeed as the freestream wind approaches the actuator disc [2],

$$a = \frac{V_i - V_t}{V_i}. \tag{2}$$

This is commonly rearranged as

$$\dot{W}_{out} = 4a(1-a)^2 \frac{\rho V_i^3 A_t}{2} = C_P\ P_{wind}, \tag{3}$$

where $C_P$=4$a$(1-$a$)$^2$ is the power coefficient and $P_{wind}$= ½$\rho V_i^3 A_t$ is considered "the power in the wind."

Differentiating the power coefficient with respect to axial induction factor and setting it equal to zero gives its maximum value as 16/27, occurring at a value of $a$ = 1/3:

$$C_{P,max} = \frac{16}{27},\ \text{when } a = 1/3. \tag{4}$$

Equation (4) represents the famous Betz limit for wind turbine performance, stating that a HAWT can deliver a maximum of just under 60% (59.3%) of the "power in the wind." Figure 2 shows the relationship of $C_P$ to $a$.

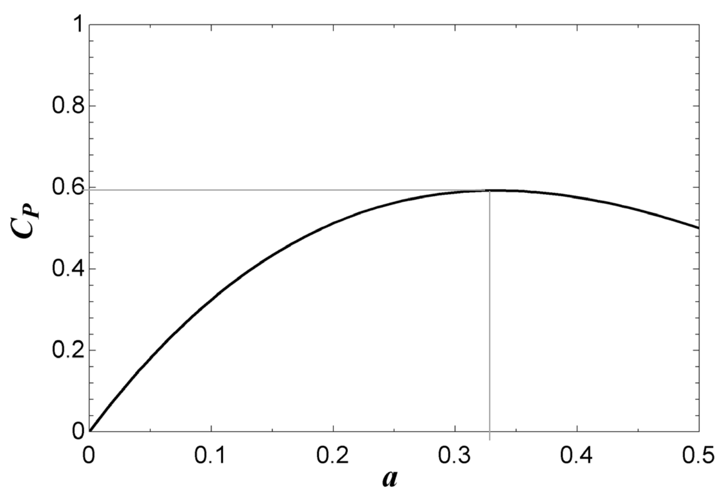


Fig. 2: Relationship of power coefficient to axial induction factor according to linear momentum theory. The maximum value of $C_P$ is 16/27 and occurs at $a$ = 1/3.

Similarly, linear momentum theory gives an expression for the thrust the wind exerts on the actuator disc as

$$F_T = C_T \frac{1}{2}\rho V_i^2 A_t, \tag{5}$$

where $C_T$=4$a$(1-$a$) is the thrust coefficient. The maximum value of thrust coefficient is $C_{T,max}$ = 1, occurring at $a$ = 1/2. Of note is that $a$ = 1/2 corresponds to a wind speed of zero at ($e$) and that maximum power and maximum thrust are not realised at the same the value of axial induction coefficient.

## 2. The Problem of Power Coefficient as Conversion Efficiency

Though the power coefficient likely serves as the most useful measure of performance for HAWTs, its interpretation as an energy conversion efficiency comes with several inconsistencies. Principal among these is what best represents the various inputs and outputs of energy. We suggest that an interpretation of power coefficient as a relative capture area bypasses such ambiguities and gives a clearer physical picture of HAWT performance. To be clear, we are not arguing that power coefficient is a poor measure of performance, but that it is not a properly defined energy conversion efficiency.

### 2.1. Two Definitions of Efficiency

In defining a conversion efficiency, we typically track the inputs and outputs of energy of a device, the efficiency being the ratio of the desired output to some input that represents the depletion of a resource. If the energy related quantities are power, then the efficiency becomes

$$\eta = \frac{P_{out}}{P_{in}}. \tag{6}$$

Implicit in employing such an efficiency is that part of the input power is not converted to the useful output, but rather to some other unusable form and/or that it is rejected elsewhere. If we consider power coefficient to be such an efficiency, then $\dot{W}_{out}$ corresponds to $P_{out}$ and $P_{wind}$ to $P_{in}$ so that

$$C_P = \frac{\dot{W}_{out}}{P_{wind}}. \tag{7}$$

Figure 2 above would suggest, then, that the maximum attainable value of efficiency for a HAWT is 16/27 and occurs at $a$ = 1/3.

In calculating power output, linear momentum theory makes use of conservation of energy applied to the stream tube of Fig. 1, the power output being the difference between the inlet and exit flows of kinetic energy:

$$\dot{W}_{out} = \dot{m}\left(\frac{V_i^2}{2} - \frac{V_e^2}{2}\right) = \rho A_i V_i \left(\frac{V_i^2}{2} - \frac{V_e^2}{2}\right), \tag{8}$$

where $\dot{m}$=$\rho A_i V_i$ is the mass flowrate of wind through the turbine. Rearranging and making use of the axial induction factor,

$$\dot{W}_{out} = \frac{\rho A_i V_i^3}{2}\left(1 - \frac{V_e^2}{V_i^2}\right) = \frac{\rho A_i V_i^3}{2} 4a(1-a). \tag{9}$$

If we consider the input flow of kinetic energy to the stream tube to be the input power, Eq. (9) cast in the form of a conversion efficiency yields

$$\eta = \frac{\dot{W}_{out}}{P_{in}} = \frac{\dot{W}_{out}}{\frac{\rho A_i V_i^3}{2}} = 4a(1-a). \tag{10}$$

Figure 3 gives a visualisation the conversion efficiency of Eq. (10).

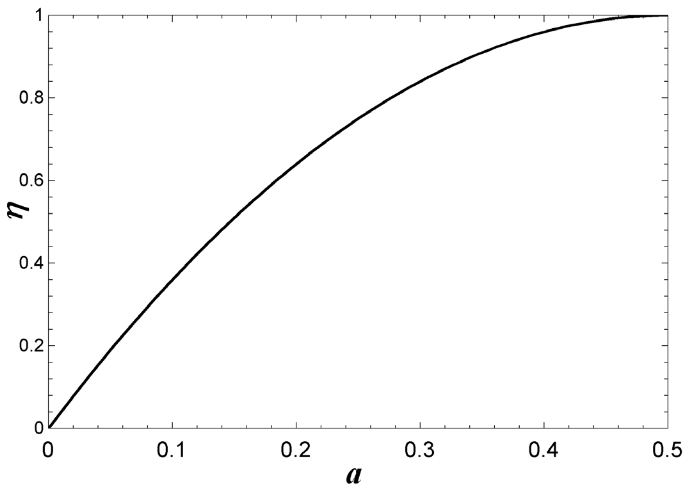


Fig. 3: Variation of energy conversion efficiency as given by Eq. (10).

Clearly Fig. 3 gives a much different picture of efficiency than Fig. 2. Rather than a maximum value of 16/27, Fig. 3 suggests that a HAWT can achieve an efficiency of one, and that that efficiency occurs at $a$=1/2, not at $a$=1/3. The reader may also recognize that the expressions for $\eta$ and the thrust coefficient $C_T$ are identical, implying that maximum efficiency and maximum thrust force are indeed achieved simultaneously.

### 2.2. Ambiguity of Inputs

The discrepancy between the two definitions for efficiency arises from the different assumed inputs for each. In Eq. (10), $P_{in}$ corresponds to the power of the upstream wind for the actual mass flowrate passing through the actuator disc, $\dot{m} = \rho A_i V_i$. On the other hand, $P_{wind}$ is the power of a hypothetical mass flow of wind at speed $V_i$ through the actuator area, $A_t$; that is, $\dot{m}_{wind} = \rho A_t V_i$. In other words, $P_{wind}$ is the power that *would be* contained in a cross-sectional area of $A_t$ if the turbine were not present.

Figure 4 shows the air flow associated with the wind turbine and allows us to contrast the different flowrates. Since the speed of the flow decreases as it approaches the actuator disc, the incompressibility of the fluid requires that the upstream area $A_i$ be less than that of the actuator disc, $A_t$. Consequently, the flow rate through the wind turbine $\dot{m}$ must be less than $\dot{m}_{wind}$, becoming smaller still with increasing values of the axial induction factor. Tracking a quantity of $\dot{m}_{wind}$ starting upstream of the actuator disc, we find that $(1\text{-}a)\dot{m}_{wind}$ makes its way through the actuator disc whereas $a\dot{m}_{wind}$ bypasses the actuator disc completely. Only in the case of zero power output, when $a$=0, does $\dot{m}_{wind} = \dot{m}$. We may therefore question whether $P_{wind}$ truly represents the power input to the turbine, since it is the power associated with $\dot{m}_{wind}$ and not with $\dot{m}$.

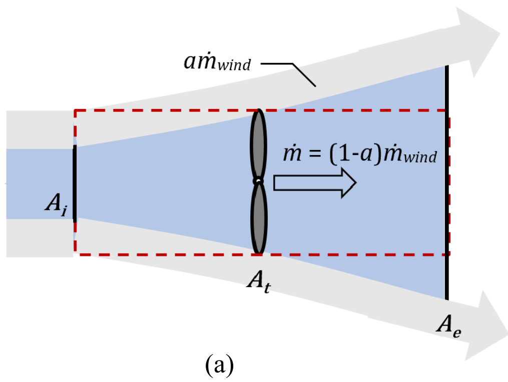


Fig. 4: (a) For any $a > 0$, a fraction of $\dot{m}_{wind}$ is deflected around the actuator disc. (b) $\dot{m} = \dot{m}_{wind}$ only when $a = 0$, which corresponds to zero power output.

Of the flow that does go through the actuator, a fraction $\eta = 4a(1\text{-}a)$ of that is converted to mechanical power, the remainder exiting with the flow downstream through $A_e$. Hence, the power coefficient accounts for two different effects, the diversion of part of $\dot{m}_{wind}$ around the actuator disc and the partial conversion of kinetic energy into power for the flow through the disc itself:

$$C_P = [\text{fraction of } \dot{m}_{wind} \text{ passing through } A_t] \cdot [\text{fraction of KE converted to } \dot{W}_{out}] \\ = [1 - a] \cdot [\eta]. \tag{11}$$

Figure 5 shows the relative proportions of $P_{wind}$ as a function of axial induction factor. The figure gives additional insight into the power coefficient as shown in Fig. 2 in that the two effects of Eq. (11) are distinguishable. The maximum power output is realized at $a$ =1/3 even though the conversion efficiency of Eq. (10) gives a value of $\eta$ = 8/9 at that point. As $a$ increases beyond 1/3, the power output decreases despite an increasing $\eta$ due to smaller inputs of $P_{in}$ delivered to the actuator disc. The trend continues until the total output of the turbine drops to 50% of $P_{wind}$ at $a$=1/2 although $\eta$ = 1.

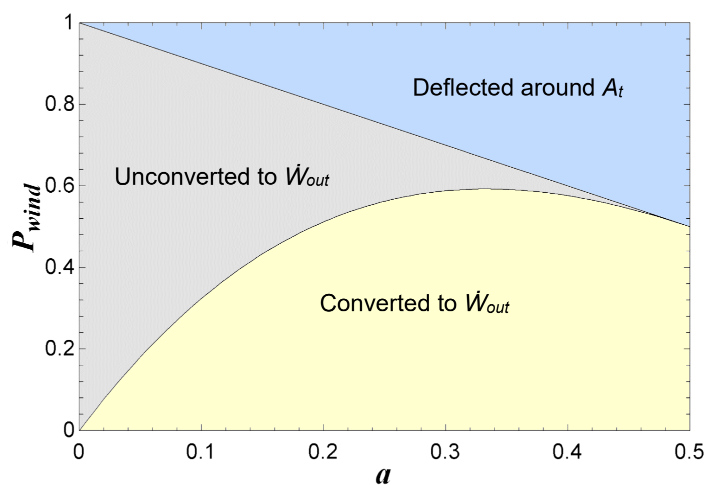


Fig. 5: Relative proportions of $P_{wind}$. The blue area is the fraction of $P_{wind}$ that is deflected around the actuator. Of the power delivered to the actuator, the yellow area is the fraction converted to power output whereas the grey area remains unconverted, ultimately exiting through $A_e$.

## 3. Reinterpretation of Power Coefficient as a Measure of Performance

Resolving the ambiguity of $C_P$ as an efficiency resides in its reinterpretation as a measure of performance. Though we may loosely consider the power coefficient as an efficiency in that it compares power output to a conveniently calculable reference, with the result (usually) being less than unity, its interpretation as an energy conversion efficiency poses several difficulties. In particular, the discussion above demonstrates that such an interpretation of $C_P$ would require us to make the rather awkward modelling assumption that a flow of wind travelling *around* the turbine—a flow that avoids the actuator disc altogether—constitutes part of the inputs and outputs of the device. Furthermore, linear momentum theory invokes isothermal and frictionless assumptions, which indicates reversible flow. Hence, there is no theoretical foundation from which to assert that 100% of the kinetic energy of a flow cannot be extracted as power. Indeed, this is what Eqs. (8) and (10) along with Fig. 3 indicate, namely, that the rate of kinetic energy input $P_{in}$ is completely converted to output power when the downstream area is very large and the speed $V_e$ becomes vanishingly small.

And so, if power coefficient does not constitute an energy conversion efficiency, what exactly does it measure?

### 3.1. Power Coefficient as Relative Capture Area

Part of the problem in defining an energy conversion efficiency for a wind turbine rests in characterizing the energy resource itself. At the root of this difficulty is the fact that there is effectively no finite amount of power that the wind contains per se, but rather what is referred to as a *power density* [5]. This power density is expressed as power per unit area,

$$P''_{wind} = \frac{\rho V_i^3}{2}. \tag{12}$$

We can therefore characterize the output of a wind turbine in terms of how much of the wind power density it captures expressed as an area,

$$\dot{W}_{out} = P''_{wind} A_{cap}, \tag{13}$$

where we refer to $A_{cap}$ as the *capture area*. Equations (3), (12), and (13) are easily rearranged to show that $A_{cap}=C_P A_t$, or

$$C_P = \frac{A_{cap}}{A_t}. \tag{14}$$

Thus, the power coefficient is best described as the *relative capture area*, a parameter that measures the amount of wind power density captured by a wind turbine relative to its size.

The idea of capture area is borrowed from ocean wave energy conversion. In wave energy devices, ocean waves are characterized via a power per unit width of wavefront, forthrightly acknowledging the difficulty in defining a discrete amount of power input to a conversion device. The equivalent width of wavefront corresponding to the power the device generates is the capture width, with that width divided by the widest linear dimension of the device forming the relative capture width. A minimum relative capture width of three or greater is cited as a guideline for the potential success of a design [2]. Also pointed out in [2] is that since the relative capture width is often greater than one, it is not helpful to consider it an efficiency.

Similarly, both capture area and relative capture area serve as useful measures of performance for wind turbines. For any turbine, we wish the capture area to be as large as possible, thereby maximizing power output. When considering relative capture area, the cross-sectional area $A_t$ is indicative of the turbine's size, and thus, its total direct cost including both capital and operational costs [6]. For two turbines with the same value of $A_{cap}$, then, we prefer the one with the larger relative capture area, as it offers us the more economical option for the same performance even if it operates at a smaller conversion efficiency.

Such an interpretation offers several advantages. Primarily, it avoids the ambiguity of what constitutes a power input to the device by more cleanly identifying the power in the wind as a density. We may also argue that it is the preferred measure of performance over an energy conversion efficiency in that it better aligns with the objective of the device. As an illustration of the latter point, we may consider the contrast between fuel economy and thermal efficiency in motor vehicles. In internal combustion engines, thermal efficiency is often highest at full throttle, indicating that the largest conversion of chemical energy into mechanical power occurs at the vehicle's top speeds. Given that air drag scales with the square of vehicle speed, however, operating at very high speeds leads to smaller distances travelled for the same amount of fuel consumed. Hence, if our objective is to travel as long a distance as possible per unit energy consumed, we may be willing to forgo always operating at the engine's highest thermal efficiency [7].

Furthermore, since $C_P$ is not a conversion efficiency, we need not limit its value to between zero to one. This is an important point when we consider wind turbine designs that seemingly surpass the Betz limit, such as HAWTs that include shrouds as outlined in [8]-[10]. The shrouds, usually in the form of diffusers, increase pressure drop across the actuator disc and thus increase mass flow. This results in power coefficients that are 2-5 times larger than traditional designs and therefore values that sometimes surpass unity. In [11] it is suggested that although the cited power coefficients are legitimate, they cease to be efficiencies for shrouded turbines. A correction to the power coefficient that effectively adjusts $P_{wind}$ to account for the increased mass flow is suggested, thereby reinstating the power coefficient's alleged status as conversion efficiency, and once again recognizing the Betz limit as its maximum value.

If we abandon the idea of power coefficient as conversion efficiency to begin with, however, such adjustments become moot—power coefficient as relative capture area becomes the common measure of performance for both shrouded and unshrouded designs, and for all other wind turbines as well, including vertical axis designs and drag machines. For a wind turbine of a given size, maximizing the relative capture area will always maximize power output, whatever the energy conversion mechanism it employs and the efficiency thereof.

## 4. Conclusion

In this paper we have shown that the power coefficient for horizontal axis wind turbines is best thought of as a relative capture area, a parameter comparing the equivalent area of the wind power density captured by a turbine relative to the turbine's size. The interpretation avoids the ambiguities associated with the power coefficient's interpretation as an energy conversion efficiency, better aligns with the objectives of operating wind turbines, and also serves as a common measure of performance for all wind turbines regardless of construction and operating principle.

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