Proceedings of the 7th World Congress on Mechanical, Chemical, and Material Engineering (MCM'21)
~~Prague, Czech Republic~~ Virtual Conference – August, 2021
Paper No. ICMIE 114
DOI: 10.11159/icmie21.114

Steady-State Creep of a Long Narrow Rectangular Membrane inside a Low Rigid Matrix with a Variable Transverse Pressure

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Abstract - The problem of the steady-state creep of a long narrow rectangular membrane inside a rigid matrix with a proportional dependence of the transverse pressure on time is studied. It considers a long rigid matrix of rectangular section, in which the height is no more than half the width. Two variants of the conditions of contact between the membrane and the matrix are considered – ideal sliding and adhesion. In this paper, three stages of membrane creep are investigated. At the first stage, the membrane is deformed under the steady creep conditions up to the moment it touches the transverse wall of the matrix. The second stage ends when the membrane touches the longitudinal walls of the matrix. At the third stage, the membrane contacts the matrix along the transverse and longitudinal sides. The calculation is carried out until the time of almost complete adhesion of the membrane to the matrix. A comparison is made of these time values under different contact conditions and levels of proximity of the membrane to the matrix.

***Keywords*:** long narrow membrane, steady-state creep, low rigid matrix, transverse pressure, ideal sliding, adhesion.

1. Introduction

This paper is devoted to an analytical study of steady-state creep in constrained conditions of a long narrow rectangular membrane fixed along the long sides and loaded with uniform transverse pressure q, which varies with time t according to a given law. The solution to this problem in free conditions in various physical and geometric situations is given in the monographs by L.M. Kachanov [1], F.K.G. Odqvist [2], N.N. Malinin [3], etc.

Of particular interest is the study of the creep of the considered membrane inside a rigid matrix. In monographs [3, 4], a cycle of creep problems of such a membrane inside a rigid matrix is studied; matrices of various shapes are considered - wedge-shaped, curved and rectangular.

2. Problem Statement

In this paper, the steady-state creep of a membrane of thickness H_0 inside a rigid rectangular matrix is investigated. The width, length, and height of the matrix are, respectively $2a$, L, b. The ratio of the matrix height to half the width satisfies the inequality $\frac{b}{a} \leq 1$ (fig.1). The width of the membrane is $2a$ and its length L satisfies the inequality $\frac{2a}{L} \ll 1$.

Here, the proportional dependence of the transverse pressure q on time t is considered.

$$q = q_0 k \frac{t}{t_0}.$$

To describe the deformation of the membrane at $t > 0$, a power-law model of the steady-state creep of the material is used

$$\frac{dp_u}{dt} = \frac{1}{t_0}\left(\frac{\sigma_u}{\sigma_0}\right)^n,$$

where σ_u and $\dot{p}_u$ are stress intensity and creep strain rates intensity respectively, σ_0, t_0 and n are constants of the corresponding dimension. These values are determined from a series of creep tests of several tensile specimens with subsequent approximation of the dependence of the steady-state creep rate on stress in the form of a power function.

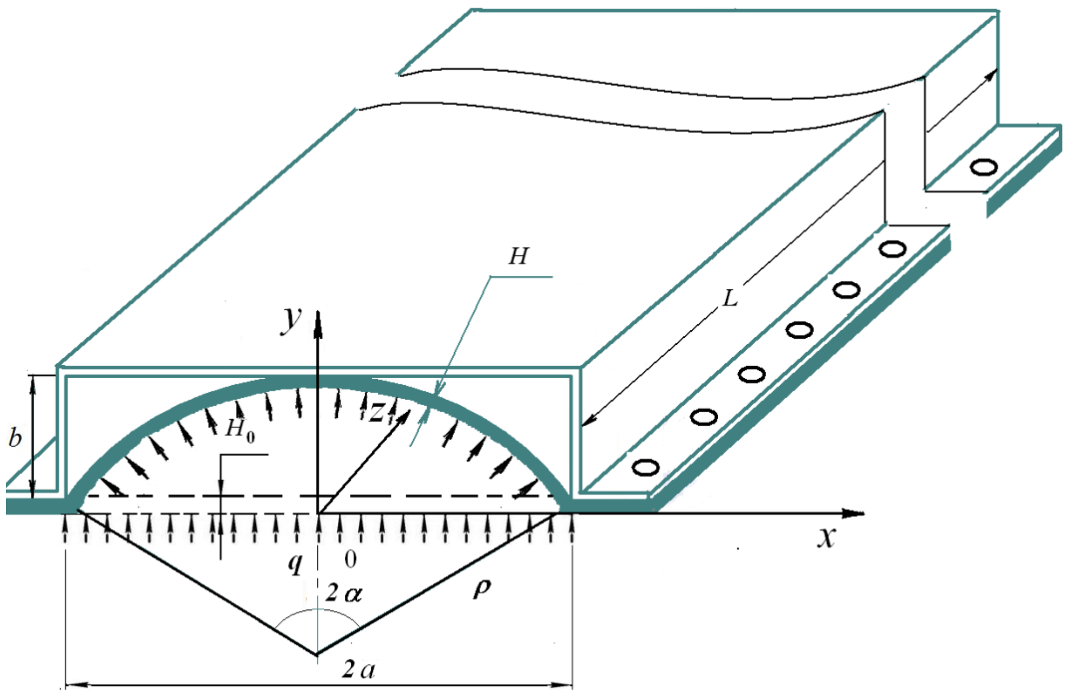

Fig. 1: General view.

The coordinates of the membrane cross-section and matrix are denominated as x and y (fig.1). At the first stage, the membrane, which is flat in the initial state, is deformed under the action of pressure q acquiring the shape of an open circular cylindrical shell with a central angle 2α. In this case, the membrane is deformed under conditions of steady creep up to the contact with the transverse wall of the rigid matrix; it can be shown that the opening angle of the membrane at the end of this stage is

$$2\alpha = 2\arcsin\left(\frac{2ab}{a^2 + b^2}\right).$$

When modeling the stress-strain state of the membrane, the principal stresses (radial σ_{rr}, circumferential $\sigma_{\theta\theta}$ and axial σ_{zz}) and the corresponding components of the creep strain tensor $p_{rr}, p_{\theta\theta}$ and p_{zz} are considered.

Considering the membrane element, taking the stresses in the element uniformly distributed over the thickness and writing down the equilibrium equations in projections onto the normal and tangent, it is obtained:

$$\sigma_{\theta\theta} = \frac{q\rho}{H}, \ d(\sigma_{\theta\theta}H) = 0, \qquad (1)$$

where ρ - radius of the median surface curvature, H – membrane thickness.

Therefore,

$$\sigma_{\theta\theta}H = const. \qquad (2)$$

Comparing (1) and (2), it is concluded that in the case of uniform pressure, the middle surface of the membrane during its deformation is part of the surface of a circular cylinder with the opening angle 2α.

3. Free Deformation of the Membrane under Creep Conditions (First Stage)

Dimensionless variables are introduced as follows

$$\bar{q}=\frac{q}{q_0},\ \bar{t}=\frac{t}{t_0},\ \bar{H}_i=\frac{H_i}{H_0},\ \bar{H}_0=\frac{H_0}{a},\ \bar{b}=\frac{b}{a},\ \bar{x}=\frac{x}{a},\ \bar{y}=\frac{y}{a},\ \bar{x}_0=\frac{x_0}{a},\ \bar{y}_0=\frac{y_0}{a} \quad (3)$$

$$\rho=\frac{\rho}{a},\ \bar{\sigma}_{ij}=\frac{\sigma_{ij}}{\sigma_0}\ \ (i,j=1,2,3)$$

x and y are the horizontal and vertical coordinates of the matrix cross-section, x_0 and y_0 are the coordinates of the points of contact between the membrane and the matrix.

Further, the dashes over all dimensionless variables (3) are omitted. In this case, the velocities are hereinafter understood as derivatives with respect to the dimensionless time.

As a connection between the components of the stress tensors and creep strain rates, the hypothesis of proportionality of the corresponding deviators (see, for example, [4]) is accepted:

$$\dot{p}_{ij}=\frac{3f(\sigma_u)}{2\sigma_u}(\sigma_{ij}-\sigma), \qquad \sigma=\frac{1}{3}\left(\sigma_{zz}+\sigma_{\theta\theta}+\sigma_{zz}\right), \quad (4)$$

$$\sigma_u=\frac{1}{\sqrt{2}}\sqrt{\left(\sigma_{rr}-\sigma_{\theta\theta}\right)^2+\left(\sigma_{\theta\theta}-\sigma_{zz}\right)^2+\left(\sigma_{zz}-\sigma_{rr}\right)^2+6\left(\sigma_{z\theta}^2+\sigma_{\theta z}^2+\sigma_{zr}^2\right)}.$$

In the considered plane deformed state, the axial creep strain rate $\dot{p}_{zz}$ is taken to be zero:

$$\dot{p}_{zz}=0. \quad (5)$$

Hereinafter, the membrane thickness at the i -th stage $\left(i=1\div 3\right)$ is denominated as $H_i(t)$

As usual, for thin-walled cylindrical shells the following equality is accepted:

$$\sigma_{rr}=0\ , \quad (6)$$

in this case, from the hypothesis of proportionality of stress deviators and creep strain rates (4), taking into account (5,6), it follows:

$$\sigma_{zz}=0.5\sigma_{\theta\theta},\ \ \sigma_u=\frac{\sqrt{3}}{2}\sigma_{\theta\theta}=\frac{\sqrt{3}}{2}\cdot\frac{q\rho}{H_0H_1}\ .$$

Considering two close deformed states of the membrane, the increment of the circumferential creep deformation is determined taking into account that the deformed state is homogeneous:

$$dp_{\theta\theta}=\frac{\left(\rho+d\rho\right)\left(\alpha+d\alpha\right)-\rho\alpha}{\rho\alpha}=\frac{d\rho}{\rho}+\frac{d\alpha}{\alpha}\ .$$

Therefore, the circumferential creep strain rate is

$$\dot{p}_{\theta\theta}=\frac{\dot{\rho}}{\rho}+\frac{\dot{\alpha}}{\alpha}=\left(\frac{1}{\alpha}-\mathrm{ctg}\alpha\right)\dot{\alpha}.$$

$$\dot{p}_{\theta\theta}=-\dot{p}_{rr}=-\frac{\dot{H}_1}{H_1}=\left(\frac{1}{\alpha}-\mathrm{ctg}\alpha\right)\dot{\alpha}, \quad \alpha\left(t=0\right)=0, \quad H_1\left(t=0\right)=1, \quad H_1\left(t\right)=\frac{\sin\alpha\left(t\right)}{\alpha\left(t\right)}.$$

The obtained expressions make it possible to represent the circumferential stress $\sigma_{\theta\theta}$ and stress intensity σ_u depending on the opening angle α:

$$\sigma_{\theta\theta}=\frac{q\rho}{H_1H_0}=\frac{q_0kt\alpha}{H_0\sin^2\alpha}\ ,\ \sigma_u=\frac{\sqrt{3}}{2}\sigma_{\theta\theta}=\frac{\sqrt{3}}{2}\frac{q_0\alpha kt}{H_0\sin^2\alpha}\ .$$

Dimensionless time is introduced:

$$\tau^n = \left(\frac{q_0}{H_0}\right)^n \cdot t^{n+1}. \qquad (7)$$

As a result of a series of transformations, it is possible to obtain the dependence $\tau_1(\alpha)$ (τ_1 means time at stage 1):

$$\tau_1(\alpha) = \left\{(n+1)\cdot\left(\frac{2}{\sqrt{3}}\right)^{n+1}\cdot\frac{1}{k^n}\int_0^\alpha\left(\frac{1}{\alpha}-\mathrm{ctg}\alpha\right)\left(\frac{\sin^2\alpha}{\alpha}\right)^n d\alpha\right\}^{\frac{1}{n}}.$$

At the end of the first stage $\left(\tau_1 = \tau_1^0\right)$, the membrane opening is $2\alpha\left(\tau_1^0\right) = 2\alpha_1$. The time moment τ_1^0 at which the first stage ends and the membrane thickness $H_1^{\circ} = H\left(\tau_1^0\right)$ are determined by the equations:

$$\tau_1^0 = \tau\left(\alpha_1 = \arcsin\left(\frac{2b}{1+b^2}\right)\right), H_1^{\circ} = H_1\left(\tau = \tau_1^0\right) = \frac{\sin\alpha_1}{\alpha_1} = \left(\frac{2b}{(1+b^2)\arcsin\left(\frac{2b}{1+b^2}\right)}\right).$$

Further, the creep of a membrane inside a rigid matrix under different contact conditions is considered.

4. Ideal Sliding of the Membrane along the Sides of the Matrix

4.1. Second Stage $(0 \le x_0 \le 1-b)$

This paper examines membrane creep within a relatively low matrix $(b \le 1)$.

Due to the axial symmetry of the membrane and the matrix, the creep of the right half of the membrane is considered at $0 \le x \le 1-b,\ \ 0 \le y \le b$.

At $\tau > \tau_1^0$, part of the membrane surface is adjacent to the transverse surface of the matrix. It can be shown that in the second stage of the creep of the matrix

$$\rho = \frac{(1-x_0)^2+b^2}{2b},\ \ \mathrm{tg}\alpha = \frac{(1-x_0)}{(\rho-b)} = \frac{2b(1-x_0)}{\left[(1-x_0)^2-b^2\right]}.$$

When studying the second stage of membrane creep, two close deformed states are distinguished: one is characterized by the length of the contact area x_0, and the other, by the length of the contact area ($x_0 + dx_0$). Using geometric relationships, the equation for the increment of the circumferential creep strain $dp_{\theta\theta}$ is obtained.

$$dp_{\theta\theta} = \frac{(\rho d\alpha + \alpha d\rho) + dx_0}{\rho\alpha + x_0} = \frac{B_1(x_0)dx_0}{B_2(x_0)} \qquad (8)$$

where $B_1(x_0)dx_0 = \rho d\alpha + \alpha d\rho + dx_0 = -\frac{1-x_0}{b}\left(\mathrm{arc\,tg}\left(\frac{2b(1-x_0)}{\left[(1-x_0)^2-b^2\right]}\right)\right)dx_0 + 2dx_0$,

$$B_2(x_0) = \rho\alpha + x_0 = \frac{\left[(1-x_0)^2+b^2\right]}{2b}\left(\mathrm{arc\,tg}\left(\frac{2b(1-x_0)}{\left[(1-x_0)^2-b^2\right]}\right)\right) + x_0.$$

Using (8), the deformed state characteristics are calculated

$$\dot{p}_{\theta\theta}=\frac{B_1\left(x_0\right)}{B_2\left(x_0\right)}\cdot\frac{dx_0}{dt},\quad d\dot{p}_u=\frac{2}{\sqrt{3}}\frac{B_1\left(x_0\right)}{B_2\left(x_0\right)}\cdot\frac{dx_0}{dt}.$$

From the incompressibility condition, taking into account (5), it is obtained: $d\dot{p}_{\theta\theta}=-d\dot{p}_{rr}$.

According to the definition $\dot{p}_{rr}$, $\dot{p}_{rr}=\frac{\dot{H}_2}{H_2}$. Therefore

$$\dot{p}_{\theta\theta}=\frac{B_1\left(x_0\right)dx_0}{B_2\left(x_0\right)dt}=-\dot{p}_{rr}=-\frac{\dot{H}_2}{H_2}.$$

As a result of a series of transformations, the dependence $\tau_2\left(x_0\right)$ can be obtained:

$$\tau_2\left(x_0\right)=\left(\left(\tau_1^0\right)^n+\left(n+1\right)\frac{1}{k^n}\int_0^{x_0}\frac{\frac{2}{\sqrt{3}}\frac{B_1\left(x_0\right)}{B_2\left(x_0\right)}}{\left[\frac{\sqrt{3}}{4}\frac{((1-x_0)^2+b^2)}{H_2\left(x_0\right)b}\right]^n}dx_0\right)^{\frac{1}{n}},\quad \tau_2^0=\tau_2\left(x_0=1-b\right).$$

4.2. Third stage $(1-b\le x_0\le 1-\Delta)$

At the third stage of the creep, the membrane touches both sides of the matrix $(1-b\le x_0\le 1-\Delta,\ 0\le y_0\le b-\Delta)$. In this case, the membrane profile is a semicircle of radius $(1-x_0)$ and the dimensions of the contact areas along the longitudinal and transverse axes are equal $(x_0-(1-b))=y_0$. Taking into account the accepted assumptions, the components of the creep strain tensor will take the form:

$$dp_{\theta\theta}=F_1(x_0)dx_0,\qquad F_1(x_0)=\frac{\left(2-0.5\pi\right)}{\left[\left(b+0.5\pi-1\right)+\left(2-0.5\pi\right)x_0\right]},$$

$$-\int_{H_2^0}^{H_3(x_0)}\frac{dH_3}{H_3}=\int_{a-b}^{x_0}F(x_0)dx_0=\ln\frac{H_2^0}{H_3\left(x_0\right)}=\ln\left[\frac{\left(b+0.5\pi-1\right)+\left(2-0.5\pi\right)x_0}{1-b+0.5\pi b}\right].$$

The end of the third stage occurs at x_0^0 satisfying the inequality $\left(1-x_0^0\right)=\Delta<<1$.

As a result of a series of transformations, the dependence $\tau_3\left(x_0\right)$ can be obtained:

$$\tau_3\left(x_0\right)=\left[\left(\tau_2^0\right)^n+\left(n+1\right)\left(\frac{2}{\sqrt{3}}\right)^{(n+1)}\cdot\left(\frac{1}{k^n}\right)\cdot\int_{1-b}^{x_0}\frac{\left(2-0.5\pi\right)dx_0}{\left[b+\left(0.5\pi-1\right)+\left(2-0.5\pi\right)x_0\right]^{(n+1)}\cdot\left(1-x_0\right)^n}\right]^{\frac{1}{n}},$$

$$\tau_3^{\ 0}=\tau_3\left(x_0^0=1-\Delta\right).$$

5. Membrane Adhesion along the Sides of the Matrix.

5.1. Second stage $(0 \le x_0 \le 1 - b)$

In the case of gradual adhesion of the membrane material to the matrix, its contact part (with variable thickness) is deformed, and the free part (with constant thickness) is a part of a circular arc. The circumferential deformation in the part of the membrane has the following form:

$$dp_{\theta\theta} = \frac{(\rho d\alpha + \alpha d\rho) + dx_0}{\rho\alpha} = \frac{B_1(x_0)dx_0}{B_3(x_0)} = -\frac{dH_2}{H_2},$$

where

$$B_3(x_0) = \rho\alpha = \frac{\left[(1-x_0)^2 + b^2\right]}{2b}\left(\operatorname{arc\,tg}\left(\frac{2b(1-x_0)}{\left[(1-x_0)^2 - b^2\right]}\right)\right).$$

As a result of a series of transformations, the dependence $\tau_2(x_0)$ can be obtained:

$$\tau_2(x_0) = \left[\left(\tau_1^0\right)^n + (n+1)\left(\frac{4}{\sqrt{3}}\right)^{(n+1)} \cdot \left(\frac{1}{2k^n}\right) \cdot \int_0^{x_0} \frac{\frac{B_1(x_0)}{B_3(x_0)}}{\left[\frac{((1-x_0)^2 + b^2)}{H_2(x_0)b}\right]^n} dx_0\right]^{\frac{1}{n}}, \quad \tau_2^0 = \tau_2(x_0 = 1-b).$$

5.2. Third Stage $(1 - b \le x_0 \le 1 - \Delta)$.

At the third stage, the creep of the membrane is characterized by touching the longitudinal and transverse sides of the matrix:

$$dp_{\theta\theta} = F_2(x_0)dx_0 = -\frac{dH_3}{H_3}, \quad F_2(x_0) = \frac{2-0.5\pi}{0.5\pi(1-x_0)} = \frac{4-\pi}{\pi(1-x_0)}.$$

The third stage ends at x_0^0 satisfying the inequality $\left(1-x_0^0\right) = \Delta << 1$.

The stress intensity is determined by the following relationship:

$$\sigma_u(x_0) = \frac{\sqrt{3}}{2}\frac{q\rho}{H_3(x_0)H_0}, \quad \rho = 1 - x_0, \quad \sigma_u(x_0) = \frac{\sqrt{3}}{2}\frac{q}{H_0}\cdot\frac{(1-x_0)}{H_3(x_0)}.$$

$$H_3(x_0) = \frac{2b}{(1+b^2)\arcsin\left(\frac{2b}{1+b^2}\right)}\exp\left[-\int_0^{1-b}\frac{B_1(x_0)}{B_3(x_0)}dx_0\right]\cdot\left(\frac{b}{1-x_0}\right)^{\frac{4-\pi}{\pi}}.$$

The intensity of the creep strain rates is

$$\dot{p}_u = \frac{2}{\sqrt{3}}\frac{(4-\pi)}{\pi}\cdot\frac{1}{(1-x_0)}\frac{dx_0}{dt}.$$

As a result of a series of transformations, the dependence $\tau_3(x_0)$ can be obtained:

$$\tau_3(x_0)=\left[\left(\tau_2^0\right)^n+\left(\frac{2}{\sqrt{3}}\right)^{(n+1)}(n+1)\cdot\left(\frac{1}{k^n}\right)\int_{1-b}^{x_0}\left(\frac{4-\pi}{\pi}\right)\cdot\frac{\left(H_3(x_0)\right)^n}{(1-x_0)^{(n+1)}}dx_0\right]^{\frac{1}{n}}.$$

The time of the almost complete adhesion of the membrane to the matrix is determined by the expression:

$$\tau_3^0=\tau_3\left(x_0=x_0^0\right).$$

6. Results Analysis.

As an example, there is considered the creep of a membrane inside a matrix with the height $b=0.5,$ with the following selected parameters:

$\mathrm{n}=5, \mathrm{k}=0.5, 1.0, 1.5, \Delta=0.01, 0.001, 0.0001.$

Table 1. The time values τ_1^0, τ_2^0 and τ_3^0 at the end of each stage in case of ideal sliding ($i=1-2$) and adhesion ($i=3-4$).

i	b	k	τ_1^0	τ_2^0	$\Delta=0.01$ τ_3^0	$\Delta=0.001$ τ_3^0	$\Delta=0.0001$ τ_3^0
1	1	0.5	1.94	1.94	37.8	238	1501
		1.0	0.971	0.971	18.9	119	751
		1.5	0.647	0.647	12.6	79.3	500
2	0.5	0.5	1.35	2.67	53.5	336	2120
		1.0	0.673	1.33	26.8	168	1060
		1.5	0.449	0.888	17.8	112	707
3	1	0.5	1.94	1.94	36.6	195	1042
		1.0	0.971	0.971	18.3	97.7	521
		1.5	0.647	0.647	12.2	65.1	347
4	0.5	0.5	1.35	2.62	54.1	289	1539
		1.0	0.673	1.31	27.1	144	769
		1.5	0.449	0.874	18.1	96.2	513

The time values τ_1^0, τ_2^0 and τ_3^0 at the end of each stage are given in the table for ideal sliding and adhesion ($i=1-2$ and $i=3-4$ respectively). Figure 2 shows the dependence of the angle $\alpha(\tau_1)$ in the case of free deformation, Figure 3 shows the dependence on time $x_0(\tau_3)$ in the case of ideal sliding (solid curve) and the dependence $x_0(\tau_3)$ in the case of adhesion (dashed curve) at $\Delta=0.0001$.

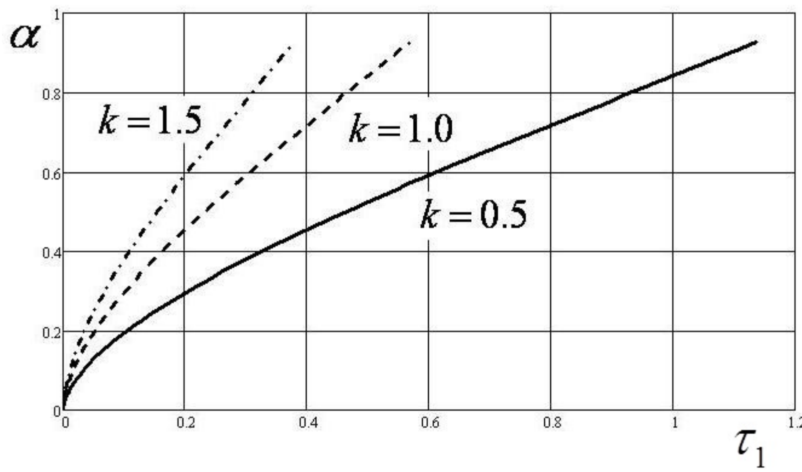

Fig. 2. First stage of deformation.

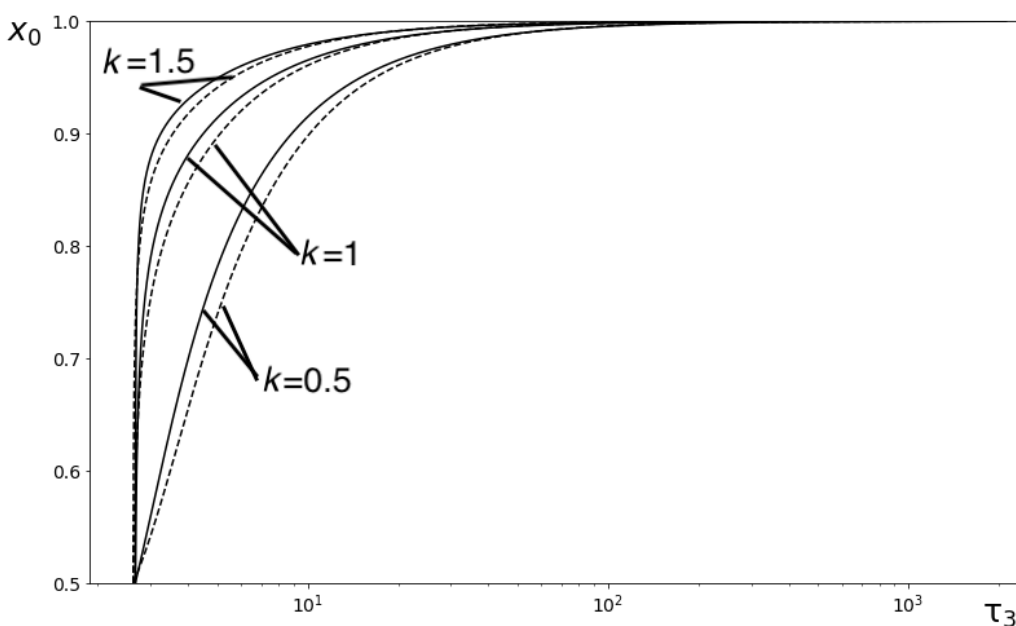

Fig. 3. Dependences $x_0(\tau_3)$ in the case of ideal sliding (solid line) and adhesion (dashed line).

Calculations show that for given values of the parameters used, the duration of membrane deformation up to adhesion of the membrane to the matrix at $\Delta = 0.001$ with ideal sliding is shorter than with adhesion; at $\Delta = 0.001$ and $\Delta = 0.0001$ the opposite result is observed. This effect is explained by the fact that in the case of adhesion of the membrane to the matrix, the thickness of the free section of the membrane $H_3(x_0)$ during its creep decreases with time due to an increase in the x_0 value, therefore, the value of the transverse creep strain $p_{\theta\theta}$ increases and, accordingly, the time $\tau_3{}^0$ decreases.

7. Conclusion

The results of a study of the steady-state creep of a long narrow membrane inside a low rigid matrix with a proportional dependence of the transverse pressure on time are presented. Two variants of the conditions of contact between the membrane and the matrix are considered: ideal sliding and adhesion. The analysis is carried out until the time corresponding to the almost complete adherence of the membrane to the matrix. A comparison of these times is carried out for different contact conditions and levels of proximity of the membrane to the matrix. An explanation of the results obtained is given.

Acknowledgements

The authors extend their gratitude to D.D. Makhov, a student of Lomonosov Moscow State University, for participation in the calculations.

The research was performed with partial financial support of the Russian Science Foundation (project No.19-19-00062).

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