*Proceedings of the 10th International Conference on Heat Transfer and Fluid Flow (HTFF'23)*
*Brunel University, London, United Kingdom –August 06-08, 2023*
*Paper No. HTFF 204*
*DOI: 10.11159/htff23.204*

# Effect of Pulsation in Relative Vortex Strength on the Evolution of A Material Line In A Pair Of Viscous Vortices

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***Abstract*** – Vortices are part of many real flow situations. Vortices help in enhancing mixing which in turn is helpful in improving heat transfer and chemical reaction rates. Mixing is natural in turbulent flow regime. Inducing good mixing in laminar flows is a challenge. To improve mixing in laminar flows, principles of chaotic mixing can be utilized. Chaotic mixing can be induced in the presence of interacting vortices. In this present work, pair of similar viscous vortices is considered. The influence of the oscillation in the relative strength of vortices on a material line is studied. Cases of these vortices rotating in the same sense and opposite sense are studied separately. Pairs of Rankine as well as Lamb-Oseen vortices are investigated. Evolution of a material line is one of the simple yet effective ways to visualize mixing. In this work, evolution of a line of 10000 particles placed along the line joining the center of the vortices is investigated. The evolution of the material line with time is tracked by tracking position of individual particles using Runge-Kutta marching scheme of the 4th order. Mixing is quantified using a simple measure called Total Average Deviation (TAD). Investigations on varying amplitudes and frequencies of oscillations have been carried out. In all the cases considered, TAD values give considerable insights into mixing of particles in the material line. TAD curve is plotted with TAD values along the vertical axis and initial x-coordinate position along the horizontal axis. Oscillation in TAD curve indicates mixing among nearby particles. Large TAD values with no oscillation in the TAD curve indicate that there is a bulk transport of the material line without mixing among them. Low TAD values with no oscillation in the TAD curve indicate less deviation from the steady case.

***Keywords*:** Mixing, Material line evolution, Rankine vortex, Lamb Oseen Vortex, Vortex pair, pulsation, Total Average Deviation (TAD)

## 1. Introduction

Vortex dominated flows are ubiquitous in nature. The study of vortex dominated flows helps us to understand many natural phenomena and is helpful in designing various industrial processes. Vortices may appear in the form of vortex-street or as trailing vortices. They appear as coherent structures in turbulent flows. Vortices play an important role in transport of energy and mass. Vortices can be modelled as inviscid, Rankine or Lamb Oseen Vortices. The interaction among these model vortices can closely mimic some realistic flow conditions. One of the significant phenomena that is of interest when researching vortex interactions is mixing. Mixing influences physical phenomena such as heat transfer and chemical reactions. In viscous laminar flows, mixing is an arduous task. To promote mixing in highly viscous flows, one has to employ the principle of chaotic advection. Chaotic advection involves repeated stretching and folding of material lines, which creates Smale's horseshoe map that is considered as one of the chaotic maps. Chaotic advection has been studied extensively using vortex models. Vortices move under the influence of each other. Moving vortices can be modelled as a dynamical system which has been studied extensively [1]. Vortices with viscous cores such as Rankine and Lamb Oseen vortices experience core deformation when they come near each other. Two viscous vortices rotating in the same sense merge into one vortex when they approach each other [2]. If the distance between the vortex centres is more than four times the vortex core diameter, merging of vortices does not take place. Two-dimensional stationary vortices can be considered as an extreme idealisation of stirrers. There has been considerable work with regard to mixing using stationary vortices. The pioneering work of Hassan Aref [3] used the blinking vortex model to study chaotic mixing. Mixing strategies involve applying time dependent oscillations on the vortex field in order to achieve stretching and folding. Mixing enhancement using chaotic advection in laminar batch mixers has been studied by Shirmohammadi and Tohidi [4]. They varied relative rotational speed of stirrers sinusoidally to enhance mixing. Rod shaped stirrers have been used to understand mixing in viscous flows [5]. Boyland et al. [6] also considered rod stirrers to study topological features of stirring by physically moving the stirrers using different stirring protocols.There are various ways of quantifying mixing, and among them a simplest way is the material line stretching [7]. The stretching rate can be defined as

$$SR = \ln\left(\frac{L(t)}{L_o}\right) \tag{1}$$

where $L(t)$ is the length of the material line of initial length $L_o$ at time $t$. In similar way, area stretching rate may also be computed. There are various bin counting techniques used to quantify mixing [8][9]. In the present work, pairs of stationary viscous model vortices are considered and material line evolution is tracked. Pairs of co-rotating and counter-rotating Rankine and Lamb-Oseen vortex pairs are considered for investigation. A material line is placed along the line joining the vortex centres as shown in the Fig. 1. The evolution of this material line with and without imposed oscillations is investigated. Mixing is quantified using a simple technique called Total Average Deviation (TAD).

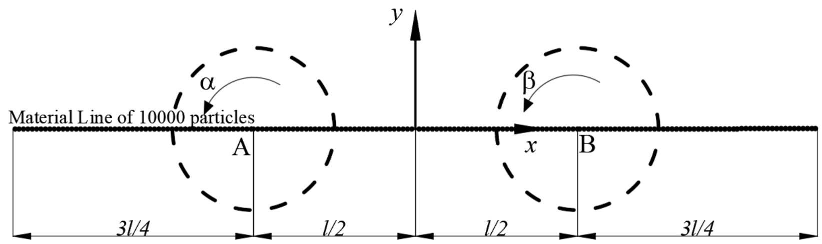



Fig. 1: Arrangement of two vortices along with the material line
(The dashed lines represent the vortex core boundaries)

## 2. Mathematical Formulation

### 2.1 Formulation of particle motion for pair of Rankine vortices

Figure 1 shows a system of two vortices. In the case of Rankine vortices, there is a uniform vorticity distribution inside the core and there is no vorticity outside the cores. The origin of the co-ordinate system is placed in the middle of the line joining centres. In this coordinate system, the non-dimensional governing equation for the particle motion influenced by Rankine vortices is given by

$$\frac{dx}{dt} = X(x,y) = \frac{\alpha}{r_1^2}(-y) + \frac{\beta}{r_2^2}(-y) \tag{2a}$$

$$\frac{dy}{dt} = Y(x,y) = \frac{\alpha}{r_1^2}(x+\frac{l}{2}) + \frac{\beta}{r_2^2}(x-\frac{l}{2}) \tag{2b}$$

where $r_1 = \sqrt{y^2 + \left(x+\frac{l}{2}\right)^2}$ , $r_2 = \sqrt{y^2 + \left(x-\frac{l}{2}\right)^2}$ .

Here, $x$, $y$ and $l$ are non-dimensionalised with respect to the radius of vortex cores. The non-dimensional vortex strengths, $\alpha$ and $\beta$ are obtained using a reference vortex strength. When the particle lies inside one of the vortex cores, the respective value of $r_1$ (or $r_2$) in Eqns. (2a) and (2b) are to be replaced by unity.

### 2.2 Formulation of particle motion for a pair of Lamb- Oseen vortices

In the case of Lamb-Oseen vortices, there is a Gaussian distribution of vorticity with maximum vorticity at the centre of the vortex. The basic non-dimensional formulation of governing equations is derived by Pai *et al.* [10] and is given by

$$\frac{dx}{dt}=u=-y\left(\frac{\alpha(1-e^{-1.25r_1^2})}{r_1^2}+\frac{\beta(1-e^{-1.25r_2^2})}{r_2^2}\right) \tag{3a}$$

$$\frac{dy}{dt}=v=\left(\frac{\alpha(x+2)(1-e^{-1.25r_1^2})}{r_1^2}+\frac{\beta(x-2)(1-e^{-1.25r_2^2})}{r_2^2}\right) \tag{3b}$$

### 2.3 Governing equations for the strength oscillations

The influence of oscillation in the strength of one vortex (B) in a two-vortex configuration (Fig. 1) of Rankine and Lamb-Oseen vortices is studied by comparing it with the steady case. The governing equations for the motion of fluid particles in field of vortices with the oscillating strength of vortex B for a pair of Rankine vortices is given by Eqns. (4a) and (4b)

$$\frac{dx}{dt}=X(x,y)=\frac{\alpha}{r_1^2}\left[-(y)\right]+\frac{\beta+A\sin(ft)}{r_2^2}\left[-(y)\right] \tag{4a}$$

$$\frac{dy}{dt}=Y(x,y)=\frac{(\alpha)}{r_1^2}(x+2)+\frac{(\beta+A\sin(ft))}{r_2^2}(x-2) \tag{4b}$$

where $r_1=\sqrt{y^2+(x+2)^2}$ and $r_2=\sqrt{y^2+(x-2)^2}$ .The governing equations for the motion of particle under the influence of a pair of Lamb-Oseen vortices with the oscillation of strength of vortex B is given by Eqns. (5a) and (5b).

$$\frac{dx}{dt}=-y\left(\frac{\alpha(1-e^{-1.25r_1^2})}{r_1^2}+\frac{(\beta+A\sin(ft))(1-e^{-1.25r_2^2})}{r_2^2}\right) \tag{5a}$$

$$\frac{dy}{dt}=\left(\frac{\alpha(x+2)(1-e^{-1.25r_1^2})}{r_1^2}+\frac{(\beta+A\sin(ft))(x-2)(1-e^{-1.25r_2^2})}{r_2^2}+\frac{\gamma x(1-e^{-1.25r_2^2})}{r_2^2}\right) \tag{5b}$$

where $r_1=\sqrt{y^2+(x+2)^2}$ and $r_2=\sqrt{y^2+(x-2)^2}$ .

### 2.4 Total Average Deviation (TAD)

To compare the influence of oscillations on vortex systems, a measure called Total Average Deviation (TAD) is used. TAD is a measure of deviation of the particle position in the oscillating case from the non-oscillating one. At a particular time *t*, if the position of the particle in the case where no pulsation is superimposed is given by $(x(t),\ y(t))$ and the particle position with pulsation is $(xp(t),\ yp(t))$ , then the deviation from periodic path is given by

$$D=\sqrt{(x(t)-xp(t))^2+(y(t)-yp(t))^2} \tag{6}$$

Total Average Deviation (TAD) for *n* samples taken along the particle path is given by

$$TAD=\frac{\sum D}{n} \tag{7}$$

### 2.5 Numerical Integration

In order to calculate the motion of particles, the governing equations have been integrated using the fourth order Runge-Kutta Method (RK4) for each fluid particle. The time step is chosen as 0.01 that corresponds to nearly closed path in non-oscillating stationary vortices. The integration has been performed using MATLAB software.

## 3. Rankine Vortices

The evolution of the material line for the case of $\alpha = 1$ and $\beta = 1$, until a non-dimensional time $T = 100$ without any imposed strength oscillations is shown in Fig. 2. It may be observed that the material line is stretched continuously forming a lemniscate. The position of the particles in the material line at $T = 100$ is used as a reference for calculating the TAD100, when strength oscillations are imposed. The particles inside the vortex cores rotate in a circular motion and the center equilibrium point in the language of dynamical systems [11] lies inside the vortex cores. The center of the lemniscate acts like a saddle equilibrium point. The particles have very slow velocities near saddle or center equilibrium points.

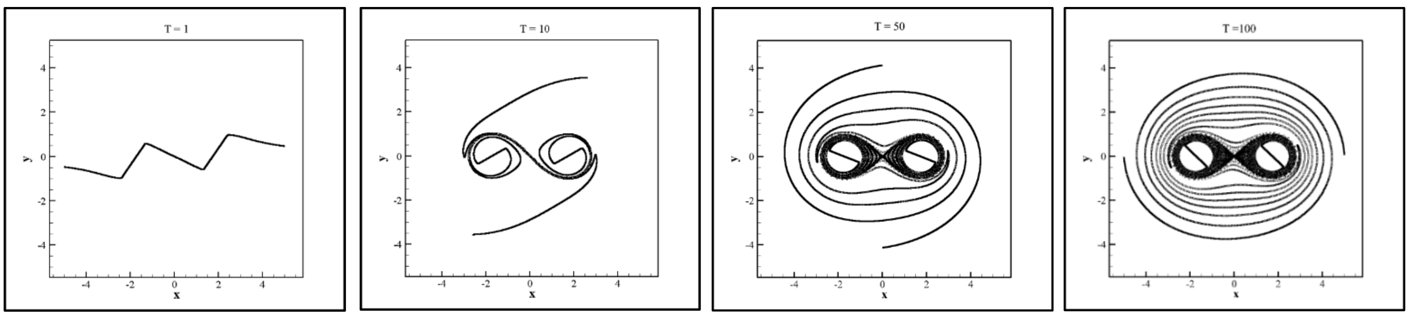



Fig. 2: Evolution of material line (Poincaré section) with time for base system having $\alpha = \beta = 1$, without any imposed oscillation

The material line positions when vortex strength oscillations are imposed at various frequencies, are presented in the Fig. 3. Figure 4 represents the corresponding values of TAD100 for each case. Well mixed regions can be observed for low amplitude $(A = 0.2)$ and low frequency $(f = 0.1)$ oscillation near the origin of the coordinate system in Fig. 3. Oscillations in TAD100 values are observed in the TAD100 plot in the Fig. 4 corresponding to these regions. In addition, it may be observed that particles on the far left and far right are considerably at different positions compared with the steady case, but mixing is not observed. This deviation is well represented with large values of TAD100. The fact that there is no oscillation in the TAD100 values may indicate poor mixing in this region. Particles inside the vortex core-A are very close to the positions in the non-oscillating case. This is reflected well with low TAD values for the corresponding locations. In the vortex B, since the material line has a different tilt compared with the non-oscillating case, the TAD100 values correspondingly show higher values for particles away from the center. For amplitude $A = 0.2$ and frequency $f = 1.0$, there is a marked oscillation of the material line inside the cores (Fig. 3). This is reflected by corresponding oscillations in the TAD values in Fig. 4. The paths on left extreme and right extreme have positions close to non-oscillating case and hence the values of TAD100 are very low. In the case of $A = 1.0$ and $f = 0.1$, the non-mixing inside the vortex core A and deviation of other locations are captured well. In case of $A = 1.0$ and $f = 1.0$ the well mixed regions are represented well by large oscillations in TAD100 values.

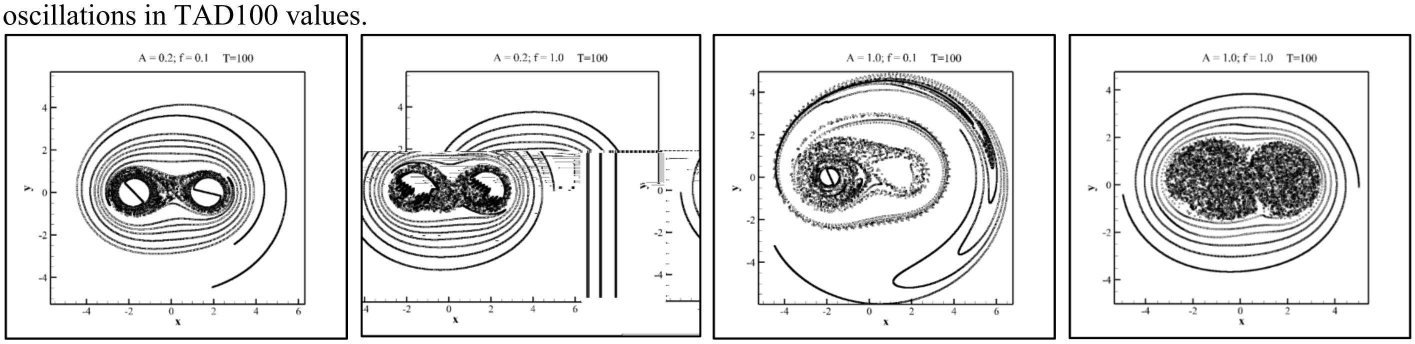



Fig. 3: State of the material line at time $T = 100$, for co-rotating Rankine vortex system ( $\alpha = \beta = 1$ ) with imposed strength oscillations of various amplitude and frequency oscillations as indicated.

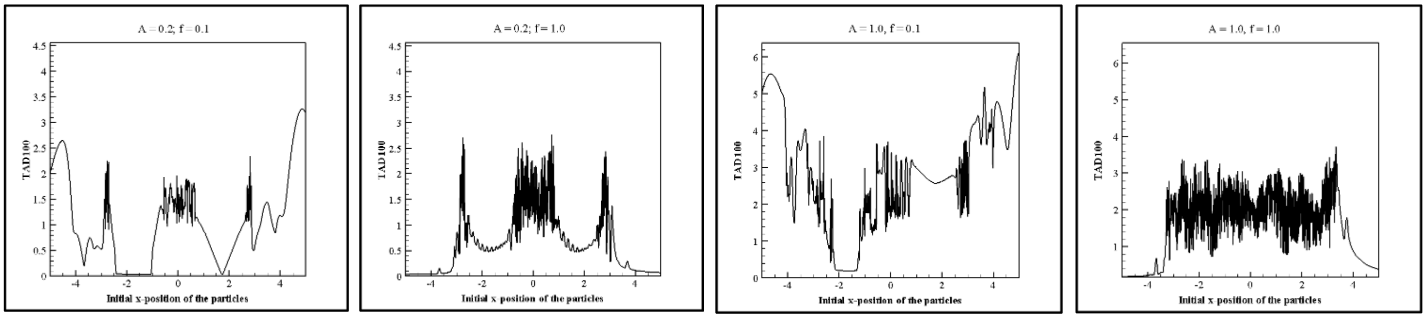
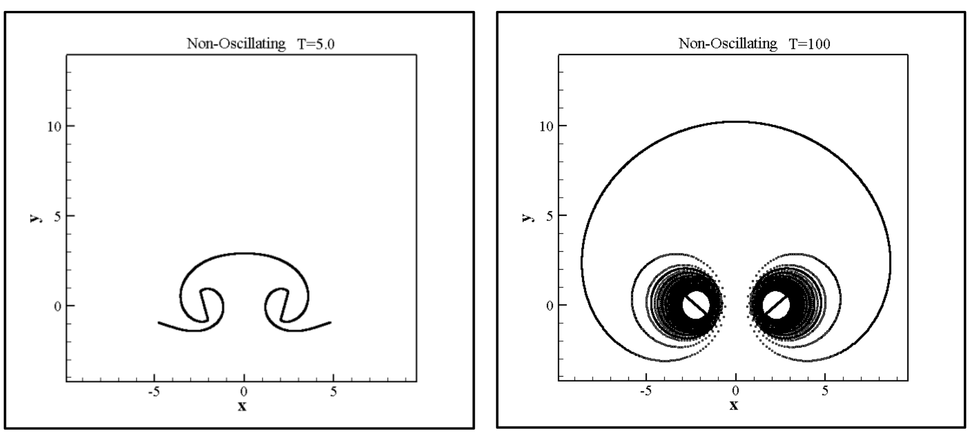



Fig. 4: Total Average Deviation (TAD) at $T = 100$, for co-rotating Rankine vortex pair, indicated as TAD100 corresponding to the imposed oscillations of amplitude and frequency as indicated.

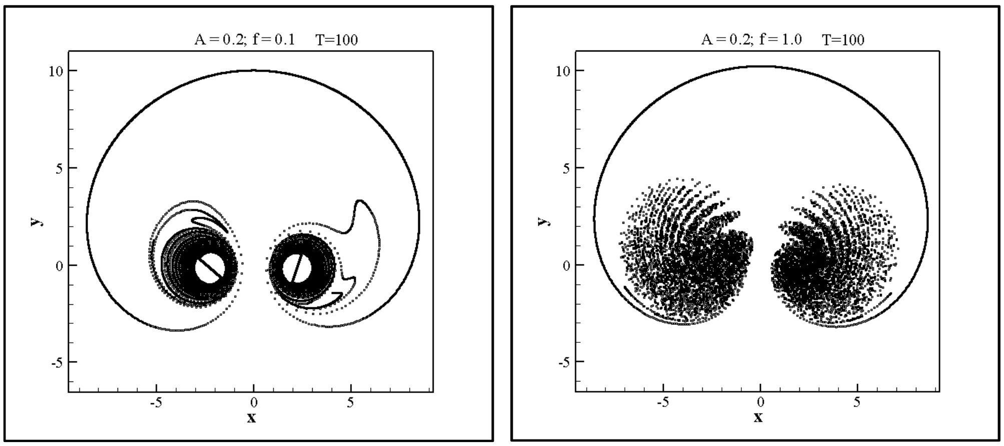
The state of the material line for the counter rotating vortex case ($\alpha = 1, \beta = -1$), after evolving for a non-dimensional time duration $T = 5$ and $T = 100$ is shown in the Fig. 5. It is observed that the particles in near the origin of the co-ordinate system, moves in the positive y-direction continuously. The part of the material line inside the vortex cores remained unchanged while other parts of the material line gets wrapped around the vortex cores. The location of the particles that are part of the material line at $T = 100$ is taken as the reference to calculate the TAD measure.




(a) (b)

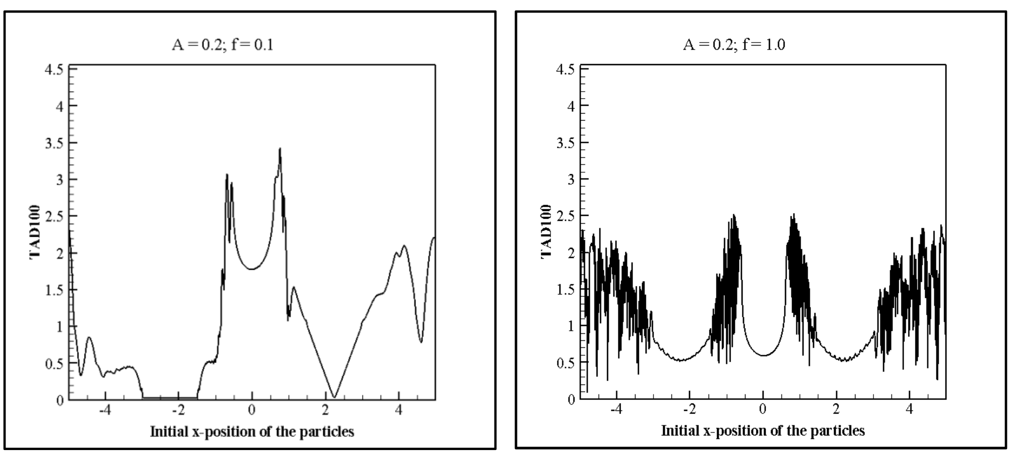
Fig. 5: Evolution of material line (Poincaré section) with time for counter-rotating Rankine vortex base system ($\alpha = 1, \beta = -1$), without any imposed oscillation for $T = 50$ and $T = 100$

The state of the material line with various imposed oscillations is shown in the Fig. 6. The corresponding TAD100 variations are shown in the Fig. 7. For a low amplitude $(A = 0.2)$ and low frequency $(f = 0.1)$ case, no much mixing is observed (Fig. 6). This is represented well with no oscillations in the TAD100 values in corresponding frequency and amplitude case in the Fig. 7. For low amplitude $(A = 0.2)$ and high frequency $(f = 1.0)$ case good mixing is observed for all regions except the region corresponding to the outer enveloping material line. Since the enveloping material line corresponds to the particles near the origin, relatively low TAD100 values are observed for particles in the Fig. 7. For a naked eye, it seems that the particles initially inside the vortex cores are well mixed, but the corresponding TAD100 values show that these particles although have moved much away from their positions, they are not well mixed among themselves. All the well mixed regions are marked with oscillations in TAD100 values.




(a) (b)

Fig. 6: Total Average Deviation (TAD) at $T = 100$, indicated as TAD100 corresponding to the imposed oscillations of amplitude and frequency as indicated.



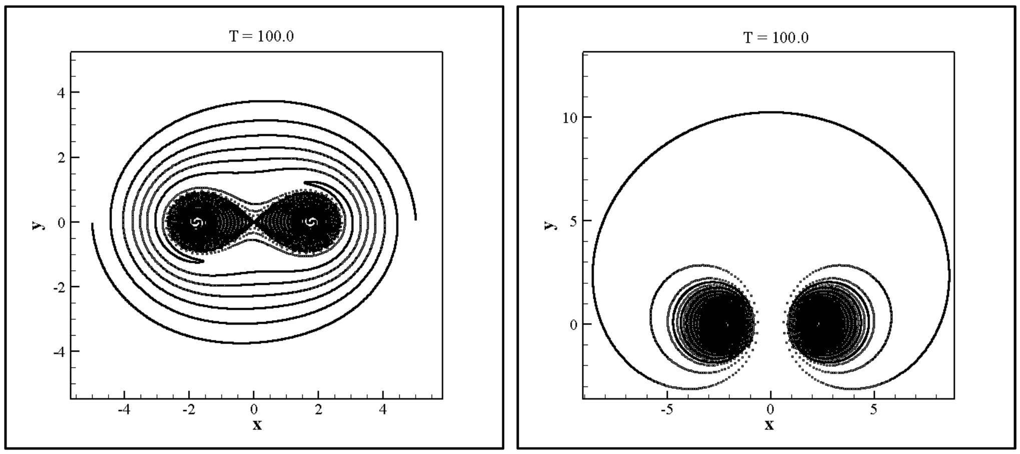

(a) (b)

Fig. 7: Total Average Deviation (TAD) at $T = 100$, indicated as TAD100 corresponding to the imposed oscillations of amplitude and frequency as indicated.

## 4. Lamb Oseen Vortices

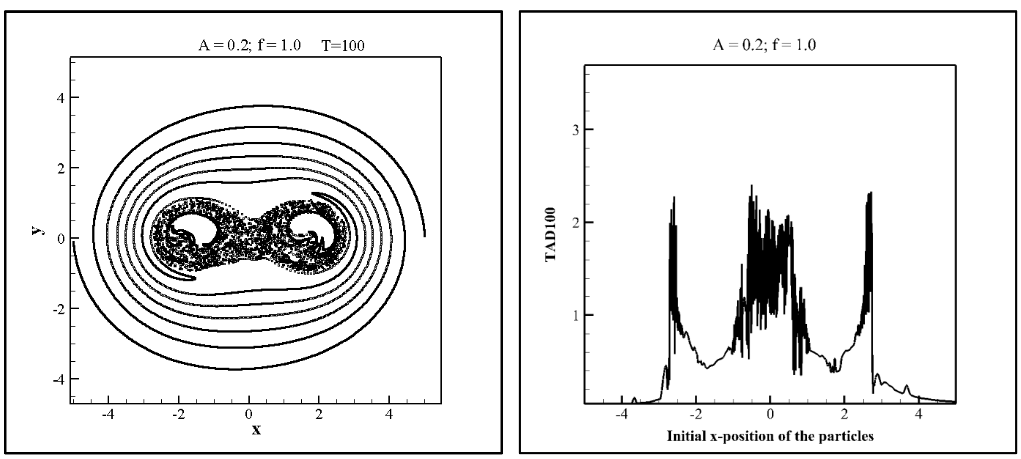
The state of the material line after non-dimensional time $T = 100$ for non-oscillating vortex strength in the system of Lamb-Oseen vortices is represented in the Fig. 8(a) for co-rotating vortices and Fig. 8(b) for counter-rotating vortices. The major difference with corresponding cases in Rankine vortices is in the region inside the core. While inside the Rankine vortex core the material line experiences rigid body rotation, the material line inside the Lamb-Oseen vortex cores stretches into spirals without mixing. The non-oscillating cases in Fig. 8(a) and Fig. 8(b) are considered as a base for calculation of TAD100 measure. Figure 9(a) represents the state of the material line for the the amplitude $A = 0.2$ and frequency $f = 1.0$

while Fig. 9(b) represents the corresponding TAD values. It is observed the well mixed regions near the saddle point (near origin) are well mixed and this is represented by oscillating TAD100 values. The regions on the far left and far right of the material line do mix well and also their positions are similar to the non-oscillating case. This is represented well in the TAD100 plot in the Fig. 9(b) as they have almost zero TAD100 values.




(a) (b)

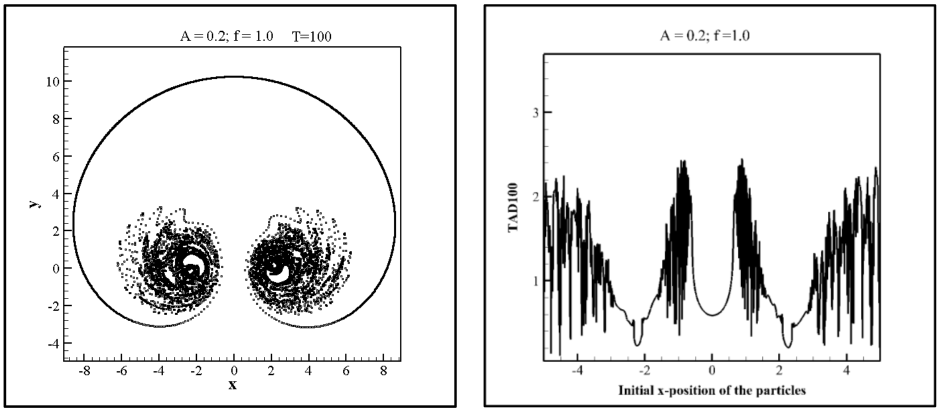
Fig. 8: State of the material line (Poincaré section) at time $T = 100$ with constant vortex strength for (a) co-rotating Lamb-Oseen vortex base system with $\alpha = \beta = 1$, and (b) counter-rotating Lamb-Oseen vortex base system with $\alpha = 1, \beta = -1$




(a) (b)

Fig. 9: State of material line (Poincaré section) with time for co-rotating Lamb-Oseen vortex base system with $\alpha = 1, \beta = 1$, for constant vortex strength

The state of the material line for the counter-rotating Lamb-Oseen vortices with oscillations of $A = 0.2$ and frequency $f = 0.1$ is shown in the Fig. 10(a) and the corresponding TAD100 values are represented in the Fig. 10(b). It is observed that the region corresponding to the outer envelope shows no oscillations in the TAD100 values where there is no mixing observed. The regions of the material line at the far left and the far right are wrapped around the vortex cores and get mixed well. This is represented by oscillations in the TAD100 values. Good mixing is also seen inside the vortex cores and that is represented well by oscillations in the TAD100 values.




(a) (b)

Fig. 10: Evolution of material line (Poincaré section) with time for co
unter-rotating Lamb-Oseen vortex base system with $\alpha = 1, \beta = -1$ , for constant vortex strength

## 5. Conclusion

The effect of strength oscillations on co-rotating and counter-rotating viscous vortices has been studied. It is observed that for low amplitude and low frequency oscillations, mixing inside the vortex cores is difficult in case of Rankine vortices. However, as the frequency increases, mixing inside the cores improves. For larger amplitudes, mixing inside the cores is achieved for lower frequencies as well. In the case of counter-rotating vortices, mixing for the regions at the far left and far right of the material line is easy compared to that of the counter-rotating vortices for the viscous vortices considered. Oscillations in the TAD100 values for the nearby initial positions indicate sensitivity to initial conditions (SIC) which is one of the defining characteristics of chaotic advection. In all the above cases considered, mixed regions are well represented by oscillations in the TAD100 values. These observations show that TAD serves as an important parameter in quantifying mixing.

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