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Boundary Value Problems of Coupled Thermoelastodynamics on a Star Graph

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Abstract: In this paper, boundary value problems of coupled thermoelasticity on a star graph are considered, for example, for analyzing the state of network structures (oil pipelines, gas pipelines, housing networks). Based on the method of generalized functions, a universal technique for solving boundary value problems on such graphs is developed. Generalized solutions of non-stationary boundary value problems are constructed under various conditions at the ends and with transmission conditions in the common node of the graph. Equations of connection for the boundary functions (displacements, stresses, temperature, heat flux) included in the generalized solutions are obtained. Regular integral representations of solutions to boundary value problems are obtained using the fundamental solution in the space of Fourier transforms. The results obtained allow modeling the action of various heat sources, including concentrated ones, using singular generalized functions. The method of generalized functions allows solving a wide class of boundary value problems, including those with local boundary conditions at the ends of the graph.

Keywords: star graph; themperature; displacement; generalized functions method; fundamental solution.

1. Introduction

There are models of uncoupled and coupled thermoelasticity. A thermoelasticity model is related if a temperature gradient is included in the motion equation in the displacements. And the equation of thermal conductivity includes a term associated with the rate of change in volumetric deformation. If you neglect this term in the equation of thermal conductivity, then the model is called uncoupled.

A great contribution to the development of the theory of thermoelasticity was made by Nowacki W. [1,2]. He justificated the models of coupled and uncoupled thermoelasticity, considered a whole class of quasi-static and dynamic thermoelasticity problems. There is detailed overview of researches on thermoelasticity in the encyclopedia [3].

Boundary value problems (BVPs) of coupled thermoelasticity belong to those classes of problems that rarely have an analytical solution even for simple structures such as beams and rods. There are few works related to the study of BVPs on thermoelastic graphs. For example, in [4], discrete Yamabe equations on star graphs were studied, resulting in a proof of the existence of nontrivial solutions. The asymptotic behavior of signal transmission of elastic and thermoelastic rods on star-shaped meshes was considered in [5] and the existence of exponential, polynomial, and slow signal attenuation rates in a thermoelastic mesh system was proven.

The authors in the works [6-8] developed the method of general function (GFM) to solve the BVPs of uncoupled and coupled thermoelastodynamics. Boundary integral equations (BIEs) were built in the space of Laplace transformations in time. The uniqueness of the set boundary problems, including taking into account shock waves, has been proven [9].

Spatially one-dimensional boundary value problems of thermodynamics using the model of uncoupled thermoelasticity were considered in [10,11].The solutions the Dirichiet problems on a star graphs were constructed for wave equation in [12,13]

The novelty of this work lies in the fact that based on the method of generalized functions, an analytical solution of the associated thermoelasticity problem on the star graph is obtained. Moreover, advantage of the method is that there is no need to re-solve the new task (Dirichlet or Neumann). In the resulting system of algebraic equations (in the corresponding matrix), only known boundary functions should be changed. So, a unified technique has been obtained for solving non-stationary initial-boundary coupled thermoelasticity on a star graph consisting of N edges.

2. The thermoelastic star graph

We consider a star graph which contains N edges (A_0, A_j) of the length L_j $(j = 1, 2, ..., N)$ with a common node A_0 (Fig. 1). On each edge $\Lambda_j = \{x \in R^1 : 0 \le x \le L_j\}$, there is a coordinate system (x,t) whose origin can be found at A_0: $x = x_1^j = 0$. The end of edge is $x = x_2^j = L_j$.

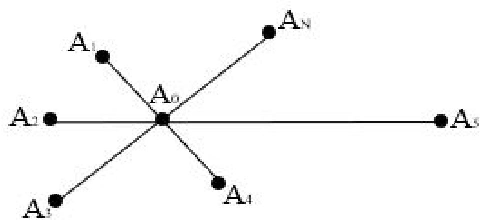

Fig1. Star graph

At Λ_j the displacement $u_j(x,t)$ and temperature $\theta_j(x,t)$ satisfies the following equations of coupled thermoelasodynamics [1,2]:

$$
\begin{aligned}
&E_j \frac{\partial^2 u_j(x,t)}{\partial x^2} - \rho_j \frac{\partial^2 u_j(x,t)}{\partial t^2} - \gamma_j \frac{\partial \theta_j(x,t)}{\partial x} + \rho_j F_{1j}(x,t) = 0, \\
&\frac{\partial^2 \theta_j(x,t)}{\partial x^2} - \kappa_j^{-1} \frac{\partial \theta(x,t)}{\partial t} - \eta_j \frac{\partial^2 u_j(x,t)}{\partial x \partial t} + F_{2j}(x,t) = 0.
\end{aligned} \tag{1}
$$

Here ρ_j, E_j are the mass density and the Young module, $\tilde{n}_j = \sqrt{E_j / \rho_j}$ is the velosities of longitudinal elastic waves in the edge; κ_j is the thermal diffusivity coefficient of the j-th edge, $\gamma_j = \alpha_j E_j$, α_j is the coefficient of thermal expansion, η_j is specific material constant, which determines the influence of the deformation rate on the change in body temperature. In uncoupled thermoelasodynamics $\eta_j = 0$.

Acting power and heat sources on the j-th edge $F_{1j}(x,t)$, $F_{2j}(x,t)$ are known functions, which belong to the class of generalized functions of slow growth on $S'(\Lambda_j)$.

The thermoelastic stress on the j-th edge is determined by the Duhamel-Newman:

$$\sigma_j(x,t) = E_j u_{j},_x - \gamma_j \theta_j \ . \tag{2}$$

Let us consider local boundary value problems of coupled thermoelasticity for a star graph, the solutions of which satisfy the initial and boundary conditions. Here and further, we denote

$$u_{j,x}=\frac{\partial u_j}{\partial x},\quad u_{j,t}=\frac{\partial u_j}{\partial t},\quad u_{j,xx}=\frac{\partial^2 u_j}{\partial x_j^2},\quad u_{j,tt}=\frac{\partial^2 u_j}{\partial t^2},\quad u_{j,xt}=\frac{\partial^2 u_j}{\partial x\partial t}.$$

We consider the following boundary value problem.

3.The statement of the first BVP of thermoelasticity on star graphs

Cauchy conditions. The initial conditions at $t = 0$ for the displacement and temperature on the graph are known:

$$u_j(x,0)=u_0{}^j(x),\quad 0\le x\le L_j,\quad u_0^j(x)=0;\quad \frac{\partial u_j(x,0)}{\partial t}=\dot{u}_0{}^j(x),\quad \dot{u}_0{}^j(0)=0,\quad 0\le x<L_j; \tag{3}$$

$$\theta_j(x,0)=\theta_0{}^j(x),\quad 0\le x\le L_j,\qquad \theta_0^j(0)=\theta_0,\quad t=0,\ j=1,...,N.$$

Dirichlet conditions. Displacement end temperature values are known at the ends of the graph:

$$u_j(L_j,t)=w_2^j(t),\text{`} \tag{4}$$

$$\theta_j(L_j,t)=\theta_2^j(t),\quad t\ge 0,\ j=1,2,...,N. \tag{5}$$

The boundary functions satisfy the following conditions:

$$u_j(L_j,t)\in C(0,\infty),\quad \theta_j(L_j,t)\in C(0,\infty). \tag{6}$$

The following *transmissions conditions* are specified in the common node A_0 of the graph:

$$w_1^j(t)-d_j w_1^1(t)=0,\quad x=0, \tag{7}$$

$$\theta_1^1(t)-\theta_1^j(t)=0,\quad t\ge 0, j=2,...,N.$$

$$\sum_{j=1}^N b_j\sigma_j(0,t)=P(t),\quad \sum_{j=1}^N \kappa_j\theta_{j,x}(0,t)=G(t),\quad x=0, t\ge 0. \tag{8}$$

Here we denote $w_1^j(t)=u_j(0,t),\ \theta_1^j(t)=\theta_j(0,t),\ \sigma_j(0,t)=E_j\,u_{j,x}(0,t)-\gamma_j\theta_j(0,t)=E_j p_1^j(t)-\gamma_j\theta_1^j(t)$.

Constants d_j, b_j are depended on the graph construction, $\mathrm{P}(t)\in C[0,\infty)$, $G(x,t)\in C[0,\infty)$.

We construct the solution of this BVP using the generalized functions method.

At first we consider the BVP of thermoelastisity at one edge of graph to define the connection betveen boundary values of displacements, temperature and their derivatives at edge ends:

$$w_1^j(t), w_2^j(t), p_1^j(t), p_2^j(t), \theta_1^j(t), \theta_2^j(t), q_1^j(t), q_2^j(t),\quad j=1,\dots,N.$$

3.1. Generalized solution on the edge of a graph

To define the connection between the boundary values of the displacement, stresses, temperature and heat flows on each elements of a graph, at first we consider the solution on an edge of graph.

Let's define $u(x,t), \theta(x,t)$ is the solution of the thermoelastisity eguations on the edge:

$$c^2u_{,xx}-u_{,tt}-\tilde{\gamma}\theta_{,x}+F_1(x,t)=0,$$
$$\theta_{,xx}-\kappa^{-1}\theta_{,t}-\eta u_{,xt}+F_2(x,t)=0,\quad 0\le x\le L. \tag{9}$$

Here $\tilde{\gamma} = \gamma / \rho$. The index of the edge is omitted.

The the displacement and temperature is known at $t = 0$:

$$u(x,0) = u_0(x), \theta(x,0) = \theta_0(x), \quad u_0(x) \in C^2\left(R^1\right), \theta_0(x) \in C^2\left(R^1\right), \quad 0 \le x \le L. \tag{10}$$

Displacement end temperature values at the ends of the edge we denote:

$$w_1(0,t) = w_1(t),\ w_2(L,t) = w_2(t), \quad 0 \le x \le L, t \ge 0. \tag{11}$$

$$\theta_1(0,t) = \theta_1(t), \theta_2(L,t) = \theta_2(t), \quad 0 \le x \le L, t \ge 0. \tag{12}$$

To determine the solution (9), we use the solution of the boundary value problem on a thermoelastic rod, previously constructed using the method of generalized functions, in [10]. It has the following form:

$$\begin{aligned}
&u(x,t)H(L-x)H(x)H(t) = -F_1(x,t) * U_1^1(x,t) - F_2(x,t) * U_1^2(x,t) + \\
&+c^2 H(t)\sum_{k=1}^{2}(-1)^{k+1}\left\{\left(p_k(t) - \tilde{\gamma}\theta_k(t)\right) \underset{t}{*} U_1^1(x-(k-1)L,t) + w_k(t) \underset{t}{*} U_{1,x}^1(x-(k-1)L,t)\right\} + \\
&+H(t)\sum_{k=1}^{2}(-1)^{k+1}\left\{\left(q_k(t) - \eta\dot{w}_k(t)\right) \underset{t}{*} U_1^2\left(x-(k-1)L,t\right) + \theta_k(t) \underset{t}{*} U_{1,x}^2\left(x-(k-1)L,t\right)\right\} - \\
&-H(L-x)H(x)\left\{\dot{u}_0(x) \underset{x}{*} U_1^1(x,t) + u_0(x) \underset{x}{*} U_{1,t}^1(x,t)\right\} + \eta \sum_{k=1}^{2}(-1)^k w_k(0) U_1^2\left(x-(k-1)L,t\right) - \\
&-H(x)H\left(L-x\right)\left[\kappa^{-1}\theta_0(x) + \eta\frac{\partial u_0(x)}{\partial x}\right] \underset{x}{*} U_1^2(x,t),
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\theta(x,t)H(L-x)H(x)H(t) = -F_1(x,t) * U_2^1(x,t) - F_2(x,t) * U_2^2(x,t) + \\
&+c^2 H(t)\sum_{k=1}^{2}(-1)^{k+1}\left\{\left(p_k(t) - \tilde{\gamma}\theta_k(t)\right) \underset{t}{*} U_2^1(x-(k-1)L,t) + w_k(t) \underset{t}{*} U_{2,x}^1(x-(k-1)L,t)\right\} + \\
&+H(t)\sum_{k=1}^{2}(-1)^{k+1}\left\{\left(q_k(t) - \eta\dot{w}_k(t)\right) \underset{t}{*} U_2^2\left(x-(k-1)L,t\right) + \theta_k(t) \underset{t}{*} U_{2,x}^2\left(x-(k-1)L,t\right)\right\} - \\
&-H(L-x)H(x)\left\{\dot{u}_0(x) \underset{x}{*} U_2^1(x,t) + u_0(x) \underset{x}{*} U_{2,t}^1(x,t)\right\} + \eta \sum_{k=1}^{2}(-1)^k w_k(0) U_2^2\left(x-(k-1)L,t\right) - \\
&-H(x)H\left(L-x\right)\left[\kappa^{-1}\theta_0(x) + \eta\frac{\partial u_0(x)}{\partial x}\right] \underset{x}{*} U_2^2(x,t).
\end{aligned} \tag{14}$$

Here Green tensor $U_i^j(x,t)$ is matrix of fundamental solutions. It is the solution of system (9) under the action of the impulse concentrated source $F = (F_1, F_2) = \delta(x)\delta(t)(\delta_1^j, \delta_2^j), \quad j = 1,2$ was obtained in [7]. The variable under the convolution sign $\left(\underset{x}{*}\right)$ shows convolution over the variable x.

The formulas (13),(14) determines the displacement and temperature inside a rod using known the displacements and temperatures at its ends. To determine unknown boundary functions (stresses, heat flows), it is necessary to construct connection equations between the boundary functions $w_1(t), w_2(t),\ p_1(t),\ p_2(t), \theta_1(t), \theta_2(t), q_1(t), q_2(t)$.

3.2. The equations of connection between boundary functions on edge of grap in Fourier transformation space in time

To define the Fourier transform of the generalized solutions (15)-(16), we use the Fourier transform of of solution (13),(14) :

$$\bar{u}(x,\omega)=-\bar{F}_1(x,\omega)\underset{x}{*}\bar{U}_1^1(x,\omega)-\bar{F}_2(x,\omega)\underset{x}{*}\bar{U}_1^2(x,\omega)+$$

$$+c^2\sum_{k=1}^{2}(-1)^{k+1}\left\{\left(\bar{p}_k(\omega)-\gamma\bar{\theta}_k(\omega)\right)\bar{U}_1^1(x-(k-1)L,\omega)+\bar{w}_k(\omega)\bar{U}_{1,x}^1(x-(k-1)L,\omega)\right\}+ \tag{15}$$

$$+H(t)\sum_{k=1}^{2}(-1)^{k+1}\left\{\left(\bar{q}_k(\omega)-i\omega\eta\, w_k(\omega)\right)\bar{U}_1^2\left(x-(k-1)L,\omega\right)+\bar{\theta}_k(\omega)\bar{U}_{1,x}^2\left(x-(k-1)L,\omega\right)\right\}.$$

$$\bar{\theta}(x,\omega)=-\bar{F}_1(x,\omega)\underset{x}{*}\bar{U}_2^1(x,\omega)-\bar{F}_2(x,\omega)\underset{x}{*}\bar{U}_2^2(x,\omega)+$$

$$+c^2\sum_{k=1}^{2}(-1)^{k+1}\left\{\left(\bar{p}_k(\omega)-\gamma\bar{\theta}_k(\omega)\right)\bar{U}_2^1(x-(k-1)L,\omega)+\bar{w}_k(\omega)\bar{U}_{2,x}^1(x-(k-1)L,\omega)\right\}+ \tag{16}$$

$$+\sum_{k=1}^{2}(-1)^{k+1}\left\{\left(\bar{q}_k(\omega)-i\omega\eta\, w_k(\omega)\right)\bar{U}_2^2\left(x-(k-1)L,\omega\right)+\bar{\theta}_k(\omega)\bar{U}_{2,x}^2\left(x-(k-1)L,\omega\right)\right\}.$$

Here

$$\bar{U}_1^j(x,\omega)=\frac{\delta_1^j\operatorname{sgn}(x)}{2(\lambda_1-\lambda_2)}\left\{i\omega\kappa^{-1}\left(\frac{\sin x\sqrt{\lambda_2}}{\sqrt{\lambda_2}}-\frac{\sin x\sqrt{\lambda_1}}{\sqrt{\lambda_1}}\right)+\left(\sqrt{\lambda_1}\sin x\sqrt{\lambda_1}-\sqrt{\lambda_2}\sin x\sqrt{\lambda_2}\right)\right\}-$$

$$-\frac{\gamma\delta_2^j\operatorname{sgn}(x)}{2(\lambda_1-\lambda_2)}\left(\cos x\sqrt{\lambda_1}-\cos x\sqrt{\lambda_2}\right),\quad j=1,2;$$

$$\bar{U}_2^j(x,\omega)=\frac{\operatorname{sgn}(x)}{2(\lambda_1-\lambda_2)}\left\{i\omega\eta\delta_1^j\left(\cos x\sqrt{\lambda_1}-\cos x\sqrt{\lambda_2}\right)-\omega^2\left(\frac{\sin x\sqrt{\lambda_1}}{\sqrt{\lambda_1}}-\frac{\sin x\sqrt{\lambda_2}}{\sqrt{\lambda_2}}\right)\delta_2^j+\right. \tag{17}$$

$$\left.+c^2\left(\sqrt{\lambda_1}\sin x\sqrt{\lambda_1}-\sqrt{\lambda_2}\sin x\sqrt{\lambda_2}\right)\delta_2^j\right\},\quad j=1,2.$$

$$\lambda_{1,2}(\omega)=\frac{\omega}{2c^2}\left\{\omega+i(\alpha+\beta)\pm\sqrt{\left(\omega+i(\alpha-\beta)\right)^2-4\alpha\beta}\right\},\ \alpha=\gamma\eta\,,\ \beta=c^2k^{-1}.$$

From Eqs (15), (16) the connection equations for boundary functions $\left(w_j(x,t),\theta_j(x,t),p_j(x,t),q_j(x,t),j=1,2\right)$ have been constructed.

Theorem 1. *The Fourier time transformants of the boundary functions on the edge satisfy the following system of linear algebraic equations in the matrix form:*

$$\mathbf{A}(\omega)\times\begin{Bmatrix}\bar{w}_1\\ \bar{p}_1\\ \bar{\theta}_1\\ \bar{q}_1\end{Bmatrix}+\mathbf{B}(\omega)\times\begin{Bmatrix}\bar{w}_2\\ \bar{p}_2\\ \bar{\theta}_2\\ \bar{q}_2\end{Bmatrix}=\mathbf{F}(\omega).$$

$$\mathbf{A}_{4\times 4}=\begin{Bmatrix}0.5 & 0 & 0 & 0\\ -\left(c^2\bar{U}_1^1,_x+i\omega\eta\bar{U}_1^2\right)_{(x=L)} & -c^2\bar{U}_1^1(L,\omega) & \left(c^2\tilde{\gamma}\bar{U}_1^1-\bar{U}_1^2,_x\right)_{(x=L)} & -\bar{U}_1^2\left(L,\omega\right)\\ 0 & 0 & 0.5 & 0\\ -\left(c^2\bar{U}_2^1,_x+i\omega\eta\bar{U}_2^2\right)_{(x=L)} & -c^2\bar{U}_2^1\left(L,\omega\right) & \left(c^2\tilde{\gamma}\bar{U}_2^1-\bar{U}_2^2,_x\right)_{(x=L)} & -\bar{U}_2^2\left(L,\omega\right)\end{Bmatrix},$$

$$\mathbf{B}_{4\times 4}=\begin{Bmatrix}\left(c^2\bar{U}_1^1,_x+i\omega\eta\bar{U}_1^2\right)_{(x=-L)} & c^2\bar{U}_1^1(-L,\omega) & -\left(c^2\tilde{\gamma}\bar{U}_1^1-\bar{U}_1^2,_x\right)_{(x=-L)} & \bar{U}_1^2\left(-L,\omega\right)\\ 0.5 & 0 & 0 & 0\\ \left(c^2\bar{U}_2^1,_x+i\omega\eta\bar{U}_2^2\right)_{(x=-L)} & c^2\bar{U}_2^1\left(L,\omega\right) & -\left(c^2\tilde{\gamma}\bar{U}_2^1-\bar{U}_2^2,_x\right)_{(x=-L)} & \bar{U}_2^2\left(-L,\omega\right)\\ 0 & 0 & 0.5 & 0\end{Bmatrix},$$

$$\mathbf{F}_{4\times 1}=-\left\{\left(F_1\underset{x}{*}U_1^1+F_2\underset{x}{*}U_1^2\right)_{x=0},\left(F_1\underset{x}{*}U_1^1+F_2\underset{x}{*}U_1^2\right)_{x=L},\left(F_1\underset{x}{*}U_2^1+F_2\underset{x}{*}U_2^2\right)_{x=0},\left(F_1\underset{x}{*}U_2^1+F_2\underset{x}{*}U_2^2\right)_{x=L}\right\}^T. \quad (18)$$

We write this system in the next matrix form:

$$\mathbf{M}(\omega)\times\ \mathbf{X}(\omega)=\mathbf{F}(\omega), \quad (19)$$

where $\mathbf{M}_{4\times 8}=\left\{\mathbf{A}_{4\times 4},\quad \mathbf{B}_{4\times 4}\right\}$ is the matrix of conection of the *boundary functions* on the edge:

$$\mathbf{X}_{8\times 1}=\left\{w_1(\omega),\ p_1(\omega),\theta_1(\omega),\ q_1(\omega),w_2(\omega),\ p_2(\omega),\theta_2\ (\omega),\ q_2\ (\omega)\right\}^T. \quad (20)$$

3.3. Resolving system of equations for determining boundary functions on a star graph

Let us return to the consideration of the first BVP problem on a thermoelastic star graph (Fig. 1). On each edge Λ_j, $j=1,...,N,$ of the star graph, we have the following system of 4 linear algebraic equations to determine 8 boundary functions:

$$\mathbf{M}^j(\omega)\times\ \mathbf{X}^j(\omega)=\mathbf{F}^j(\omega),\quad j=1,...,N. \quad (21)$$

Here j is the number of the corresponding graph edges, $\mathbf{M}^j$ is the matrix of conection of boundary functions (19) on the j-th edge which has the form:

$$\mathbf{M}^j(\omega)=\left\{\mathbf{A}^j(\omega),\quad \mathbf{B}^j(\omega)\right\}=\begin{Bmatrix}A_{11}^j & ... & \dots & A_{14}^j & B_{11}^j & \dots & \dots & B_{14}^j\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ A_{41}^j & \dots & \dots & A_{44}^j & B_{41}^j & \dots & \dots & B_{44}^j\end{Bmatrix}.$$

Its components are defined by formular (17) for $\bar{U}_i^j(x,\omega)$ with use thermoelastic parameters and the length L_j of this edge.

Thus, we have $4N$ equations for $8N$ sought boundary functions. If we add to them $2N$ Dirichlet conditions at the end of the graph (4),(5) , and $2N$ transmission conditions at the node of the graph (7),(8) , then we obtain a complete resolving system of linear algebraic equations for determining the Fourier transforms of the $8N$ sought boundary functions. Let us formulate this statement as a theorem.

Theorem 2. *The resolving system of the Dirichlet boundary value problem* (1) - (8) *on a star graph with N edges has the following form:*

$$\mathbf{W}(\omega)\times\mathbf{Y}(\omega)=\mathbf{D}(\omega),$$

and its solution is equal to

$$\mathbf{Y}(\omega)=\mathbf{W}^{-1}(\omega)\times\mathbf{D}(\omega). \tag{22}$$

Here the matrix $\mathbf{W}(\omega)$ has the dimension $8N\times 8N$. The first $4N$ rows contain the matrixes of connection of *boundary functions* for each edge of this graph $\mathbf{M}^j(\omega)$. The next $2N$ rows discribe transmission conditions at the node of the graph (7),(8) and next $2N$ rows the Dirichlet conditions (4),(5) at the ends of the graph.

The vector $\mathbf{Y}_{8N\times 1}$ contains N boundary functions $\mathbf{X}^j$ (20) which correspond to *j*-th of grapf:

$$\mathbf{Y}=\left\{\mathbf{X}^1(\omega),\mathbf{X}^2(\omega),....,\mathbf{X}^N(\omega)\right\}^T,$$

$\mathbf{D}(\omega)=\left\{\mathbf{F}^1(\omega),\mathbf{F}^2(\omega),...,\mathbf{F}^N(\omega),\mathbf{C}(\omega)\right\}^T$ is the vector which contains actions of external forces and heat flow (18) in according to j-th edge of the grapf ($4N$ component) and the vector $\mathbf{C}$.

Components of $\mathbf{C}(\omega)$ contain the right part of $2N$ transmissin conditions (7), (8) and $2N$ boundary conditions (11), (12) on the ends of graph adges.

Further, substituting $\mathbf{X}^j(\omega)=\left(w_1^j,\, p_1^j,\bar{\theta}_1^{\,j},\, q_1^j\;\, w_2^j,\, p_2^j,\bar{\theta}_2^{\,j},\, q_2^j\right)$ in the corresponding components of $\mathbf{Y}$ for *j*-edge into formulas (20), (21) we get the displacement and temperature it any its point.

So we get the values of $\bar{u}_j(x,\omega),\bar{\theta}_j(x,\omega)$ (j=1,…, N) at any edges of this star grafh.

Performing the inverse Fourier transform, we obtain originals $u_j(x,t),\theta_j(x,t)$ at any point x of this star grafh. Thus, the displacement and temperature on the star graph are determined at any time t at any edge. So BVP of coupled thermoelasodynamics for a star graph has been solved.

4. Conclusion

The method of generalized functions presented here allows not only to solve a wide range of problems with different conditions at the ends of the edges of the graph and the conditions of transmission in its common node, but can also be extended to network structures of various types. This distinguishes this method from all others that are used to solve similar problems.

The action of heat sources can be modeled by both regular and singular generalized functions under various boundary conditions at the ends of the edges.

The obtained regular integral representations of the Fourier transforms of the solutions allow to determine the displacement and temperature on each edges and at any point of the graph for stationary oscillations with a frequency ω . In the case of periodic oscillations that can be expanded in a Fourier series over time, the constructed transformations of the solutions give a solution for each harmonic of the series with a frequency $\omega = \omega_n$.

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