Proceedings of the 9th World Congress on Momentum, Heat and Mass Transfer (MHMT'24)
London, United kingdom – April 11 – 13, 2024
Paper No. ENFHT 133
DOI: 10.11159/enfht24.133

The Virtual Mass of Two Bubbles Rising in Liquids

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***Abstract* -** Equations were derived for the virtual mass coefficients C_m of two bubbles accelerating with the flow along their line of centres and perpendicular to their line of centres. The method of solution was based on potential flow. The solution proved that C_m is higher at low separation of the two side-by-side bubbles and lower at low separation of the two bubbles accelerating along their line of centres. All the derived solutions of C_m approached asymptotically the classical value of 0.5 for the single accelerating bubble. The solutions were compared favourably with other available theoretical and experimental results.

***Keywords*:** Virtual mass coefficient · Potential flow · Two bubbles · Analytical solution

1. Introduction

During bubble acceleration in liquid, work must be done in changing the kinetic energy of the liquid surrounding the bubble, which is above the work necessary to change the kinetic energy of the bubble itself. This additional work can be found by calculating the "virtual" mass of the bubble. Some authors prefer to use "added mass" instead of virtual mass.

In computing this virtual mass, it is permissible to ignore viscosity and compute the additional force on the basis of potential flow as was done by van Wijngaarden [1] and Kendoush [2].

The calculation of the virtual mass of two bubbles helps to understand the hydrodynamic interaction in the flow of bubble swarms. Kok [3] calculated the kinetic energy of two bubbles in general motion through unbounded perfect fluid by solving Lagrange's equations numerically.

Legendre et al. [4] studied numerically the three-dimensional flow past two identical spherical bubbles moving side by side by solving the full Navier-Stokes equations to get the drag and lift forces. Kendoush et al. [5] found experimentally that the virtual mass coefficient (C_m) of two solid spheres accelerating in line with their line of centers increases with spheres spacing while those moving side by side decreases with spheres spacing. It is well known in the field of particle hydrodynamics, that both types of particles (fluid and solid) have the same C_m which only depends on particle shape and type of motion (that is, linear acceleration or rotation).

The exploration of bubble coalescence phenomenon depends heavily on the exact formulation of the virtual mass effect as the downstream bubble normally accelerates to catch up with the leading bubble (Prince and Blanch [6], Kumaran and Koch [7] and Duineveld [8]). The recent communication on bubble coalescence of Hasan and Zakaria [18] did not take into account of the effects of the virtual mass.

The aim of the present study is to derive analytically, closed form equations for the virtual mass coefficients of two identical bubbles accelerating in liquids with tandem and side by side orientation.

2 Theory

2.1 Two Bubbles Rising in Line

Consider two equal-sized bubbles (A) and (B) as shown in Fig. 1, where a is the radius of both bubbles and L is the distance between their centres. Assume no deformation in the shape of bubbles as they rise. Using potential flow the following equations for the velocity of rise for bubble (A) is obtained in the Appendix as follows

$$U_A = U\left[1 + (a/L)^3\right]^{-2} \quad (1)$$

where

$U = a^2 g\rho/9\mu$ is the terminal velocity of a single bubble in an infinite medium (Batchelor, [10]), and for bubble (B)

$$U_B = U\left[1 + (a/L)^6\right]^{-2} \quad (2)$$

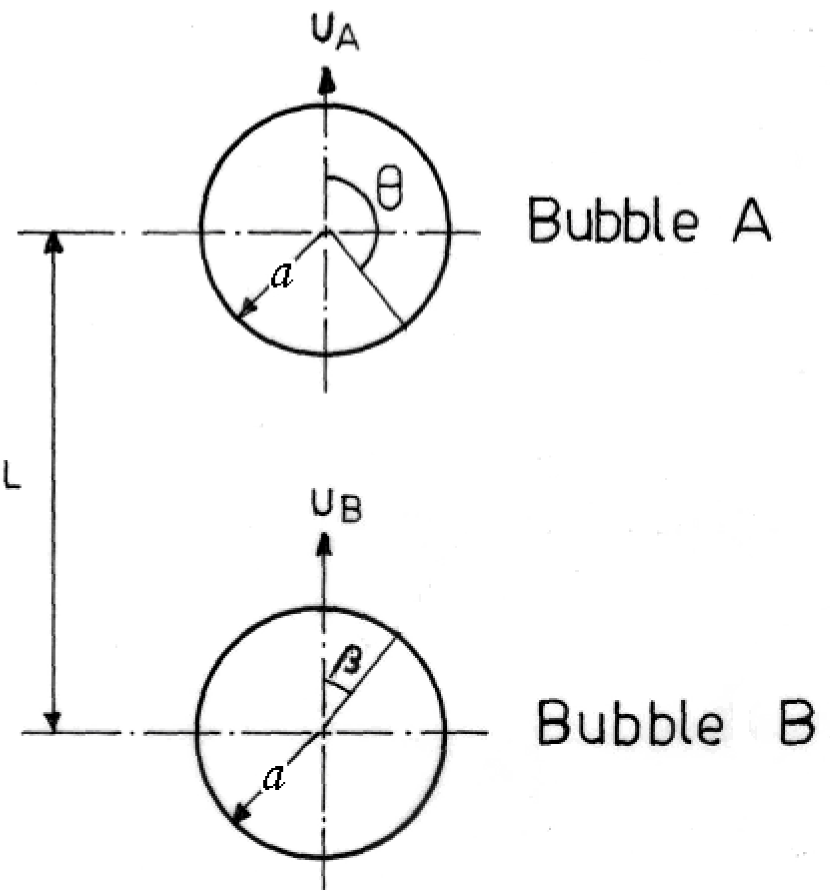

Fig. 1. Two bubbles vertically one above the other in a flow field.
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The kinetic energy of the fluid surrounding the two bubbles rising in line is given by Milne-Thomson [11] based on irrotational and inviscid flow as follows

$$KE = \frac{1}{4}M_A U_A^2 - \frac{2\pi a^3 b^3}{L^3}U_A U_B + \frac{1}{4}M_B U_B^2 \tag{3}$$

where M_A, M_B are the mass of the fluid displaced by bubble A and bubble B respectively, a and b are the radii of of bubble A and bubble B respectively. The sign of the middle term of this equation was shown positive in Milne-Thomson [11], because the spheres were moving towards each other, it was changed to negative because our spheres were moving in the same direction. For two equal-size spheres $a = b$ and $M_A = M_B = (4/3)\pi a^3 \rho$, Eq. (3) becomes the following

$$KE = \frac{1}{4}\left(\frac{4}{3}\right)\pi a^3 \rho U^2 \left[1 + \left(\frac{a}{L}\right)^3\right]^{-4} - 2\pi a^3 \rho U^2 \left(\frac{a}{L}\right)^3 \left[1 + \left(\frac{a}{L}\right)^3\right]^{-4}$$
$$+\frac{1}{4}\left(\frac{4}{3}\right)\pi a^3 \rho U^2 \left[1 + \left(\frac{a}{L}\right)^6\right]^{-4} \tag{4}$$

or, the following

$$KE = \pi a^3 \rho U^2 \left\{\frac{1}{3}\left[1 + \left(\frac{a}{L}\right)^3\right]^{-4} - 2\left(\frac{a}{L}\right)^3 \left[1 + \left(\frac{a}{L}\right)^3\right]^{-2} \left[1 + \left(\frac{a}{L}\right)^6\right]^{-2} + \frac{1}{3}\left[1 + \left(\frac{a}{L}\right)^6\right]^{-4}\right\} \tag{5}$$

or, in a more compact form

$$KE = \pi a^3 \rho U^2 G \tag{6}$$

where we called the parameter in the curly bracket of Eq. (5) by G as follows

$$G = \left\{\frac{1}{3}\left[1 + \left(\frac{a}{L}\right)^3\right]^{-4} - 2\left(\frac{a}{L}\right)^3 \left[1 + \left(\frac{a}{L}\right)^3\right]^{-2} \left[1 + \left(\frac{a}{L}\right)^6\right]^{-2} + \frac{1}{3}\left[1 + \left(\frac{a}{L}\right)^6\right]^{-4}\right\} \tag{7}$$

We shall do some manipulation in order to make this expression similar to the classical expression of kinetic energy $KE = \frac{1}{2}mU^2$, therefore Eq. (6) becomes the following

$$KE = \frac{1}{2}(2\pi a^3 \rho)U^2 G \tag{8}$$

This equation, in fact, gives the virtual mass of the two bubbles as follows

$$m = 2\pi a^3 \rho G \tag{9}$$

The virtual mass coefficient C_m is defined as the ratio of the volume of the “virtual mass” to the volume of the fluid displaced by the two bubbles, accordingly we get the following

$$C_m = \frac{2\pi a^3 \rho G}{2(4/3)\pi a^3} \tag{10}$$

or, the following after substituting for the value of G and rearranging

$$C_m = \frac{1}{4}\left[1 + \left(\frac{a}{L}\right)^3\right]^{-4} - \frac{3}{2}\left(\frac{a}{L}\right)^3 \left[1 + \left(\frac{a}{L}\right)^3\right]^{-2} \left[1 + \left(\frac{a}{L}\right)^6\right]^{-2} + \frac{1}{4}\left[1 + \left(\frac{a}{L}\right)^6\right]^{-4} \tag{11}$$

as $L \to \infty$, this equation recovers the classical value of 0.5 for the virtual mass coefficient of the single spherical bubble. This is a new equation that has not been reported before.

2.2 Two Bubbles Rising Side by Side

Fig. 2 shows two equal-sized bubbles with their equal speed of rise as given by Kendoush [11] and the Appendix as follows

$$U_A = U_B = U\left[1 + 0.5\left(\frac{a}{L}\right)^6 + \frac{1}{16}\left(\frac{a}{L}\right)^{12}\right]^{-1} \tag{12}$$

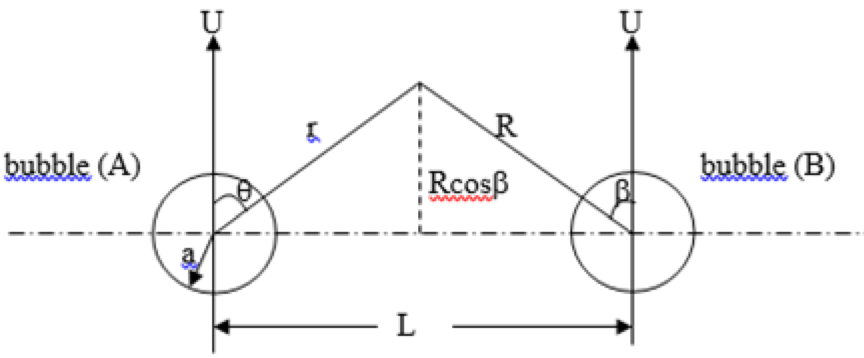

Fig.2 Two bubbles rising horizontally side-by-side in a flow field

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or, we may say

$$U_A = U_B = UK \tag{13}$$

where

$$K = \left[1 + 0.5\left(\frac{a}{L}\right)^6 + \frac{1}{16}\left(\frac{a}{L}\right)^{12}\right]^{-1} \tag{14}$$

The kinetic energy equation of Milne-Thomson [11] for the fluid around two spheres rising side by side is given as follows

$$KE = \frac{1}{4}M_A U_A^2 - \frac{2\pi a^3 b^3}{L^3}U_A U_B + \frac{1}{4}M_B U_B^2 \tag{15}$$

As we have done before in Section 2.1, for two equal-size spheres $a = b$ and $M_A = M_B = (4/3)\pi a^3 \rho$, Eq. (15) becomes the following

$$KE = \pi a^3 \rho U^2 K^2 \left[\frac{2}{3} + \left(\frac{a}{L}\right)^3\right] \tag{16}$$

From this equation, we may get the virtual mass as we have done earlier as follows

$$m = 2\pi a^3 \rho K^2 \left[\frac{2}{3} + \left(\frac{a}{L}\right)^3\right] \tag{17}$$

and the virtual mass coefficient of the two bubbles moving side by side, becomes as follows

$$C_m = \frac{3}{4}\left[1 + 0.5\left(\frac{a}{L}\right)^6 + \frac{1}{16}\left(\frac{a}{L}\right)^{12}\right]^{-2}\left[\frac{2}{3} + \left(\frac{a}{L}\right)^3\right] \tag{18}$$

as $L \rightarrow \infty$, this equation recovers the classical value of 0.5 for the virtual mass coefficient of the single spherical bubble.

3. Discussion and Validation of Results

Fig. 3 shows a comparison between the virtual mass coefficient of two bubbles rising side by side given by Eq. (18) and the virtual mass coefficient of the two bubbles rising in line given by Eq. (11). The reason of higher C_m in the side-by-side bubbles is due to the fact that the kinetic energy created in the fluid by these two bubbles is apparently greater than that created by the two bubbles rising in line. In the Appendix, it is proved theoretically that the velocity of rise of any one bubble of the two side-by-side pair is higher than the leading or the trailing bubble in the tandem orientation. Naturally, accelerated velocity leads to higher kinetic energy and higher C_m.

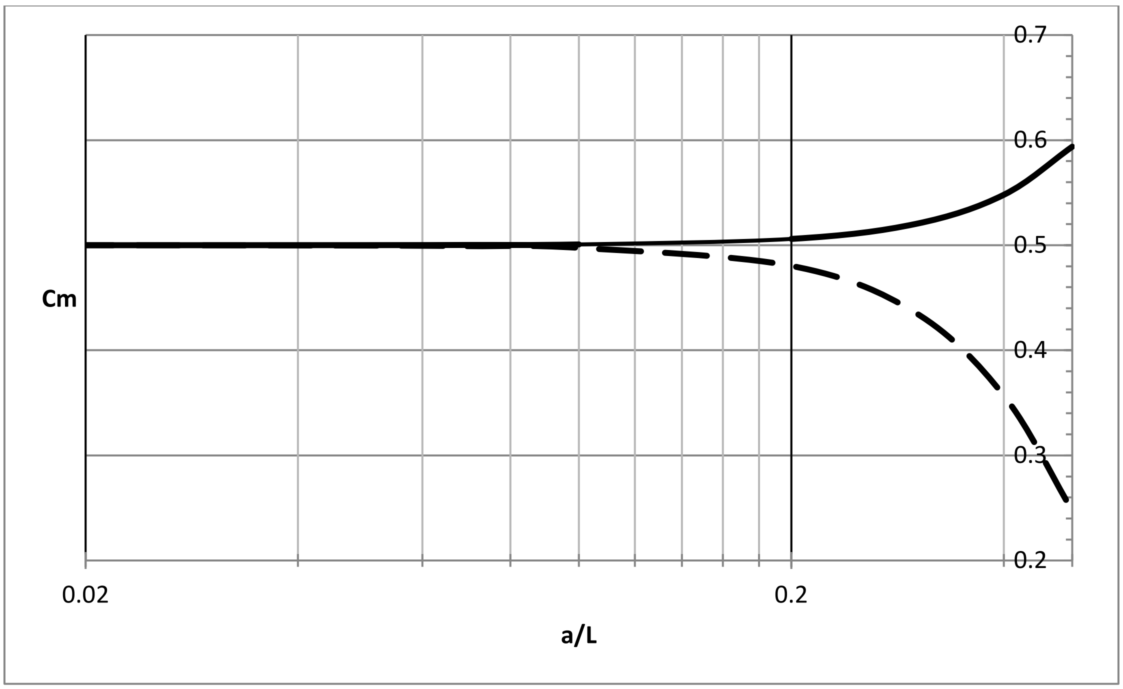

Fig. 3 Comparison between C_m of the tandem bubbles (dashed line) Eq. (11) and the side by side bubbles (solid line) Eq. (18).

Figure 4 shows a comparison between the present solution (Eq. 11), and the following equation of van Wijngaarden [1] for two bubbles rising in line

$$C_m = 0.5\left[1 - 3(a/L)^3 + 3(a/L)^6 + 9(a/L)^8 - 3(a/L)^9 + \cdots\right] \tag{19}$$

The experimental results of Kendoush et al. [5] for a particle diameter of 9.4 mm, are shown in Fig. 4, together with the numerical results of Helfinstine and Dalton [9], as well as the analytical results of Kamp et al. [13] who used the kinetic energy equation of Lamb [14] in their derivation. The increase of C_m as the two bubbles approach each other makes the solution of Kamp et al. [13] a unique one due to their assumption of the two bubbles moving towards each other. The data of Kamp et al. [13] were shown in their Table 1, which I reproduced graphically in Fig.4.

Van Wijngaarden [1] derived the following equation for C_m of two bubbles rising side by side

$$C_m = 0.5\left[1 + (3/2)(a/L)^3 + (3/4)(a/L)^6 + 3(a/L)^8 + (3/4)(a/L)^9 + \cdots\right] \tag{20}$$

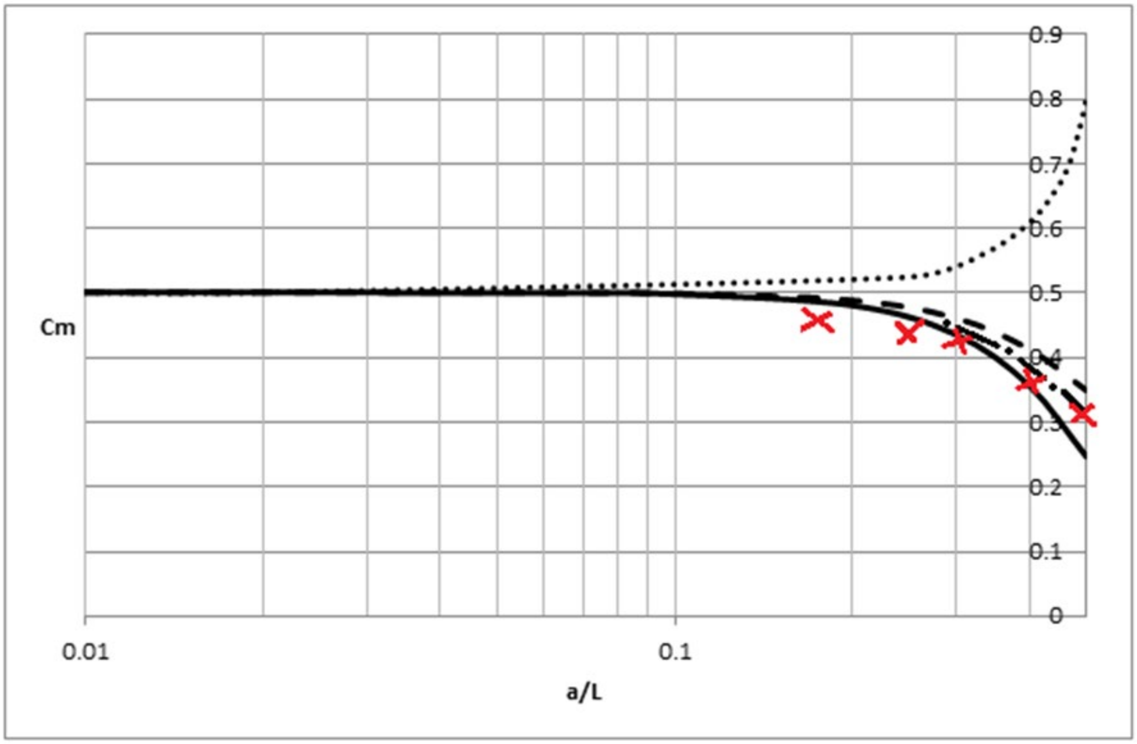

Fig. 4. Comparison between the present solution for the C_m of the two bubbles rising in line (Eq. 11), solid line, Kamp et al. [13], dotted line, van Wijngaarden [1], Eq. (19), dashed line, dash dotted line, Helfinstine and Dalton [9], and crosses, Kendoush et al. [5].

Eq. (20) is shown plotted in Fig. 5 together with the present solution of Eq. (18). Close agreement is shown between the two solutions. The reason of the slight difference between the present solution and that of van Wijngaarden [1] is that I avoided the series representation of the velocity formulas by using the viscous dissipation function of Eq. (A4), of the Appendix, where the integral was evaluated at $r = a$. Fig. 5 shows the experimental data of Kendoush et al. [5] where the agreement was not too close, but the trend of the increase at close separation is agreeable with both theories.

For further validating the present results, a comparison was made with the experimental data of Sanada [15] and Sanada et al. [16] who carefully measured sizes, separations, trajectories and velocities of two bubbles using special types of fluids. The velocity of the leading bubble in the tandem pair was modeled by the following equation of motion which incorporates the newly derived C_m of Eq. (11) and the drag force on bubble A of Eq. (A7)

$$\frac{d}{dt}\left(mC_m U_A\right) = \frac{4}{3}\pi a^3 \rho g - 12\pi\mu a U_A\left[1+\left(a/L\right)^3\right]^2 \tag{21}$$

The first term on the right side represents the buoyancy and the second term is the drag force. The mass of the bubble is neglected in Eq. (21) as well as the history (Basset term) because their contributions are considered minimum. This equation was solved by using the physical properties of Table 1. Eq. (21) was solved with the following initial conditions

$$U_A = 0 \quad \text{at } t = 0 \tag{22}$$

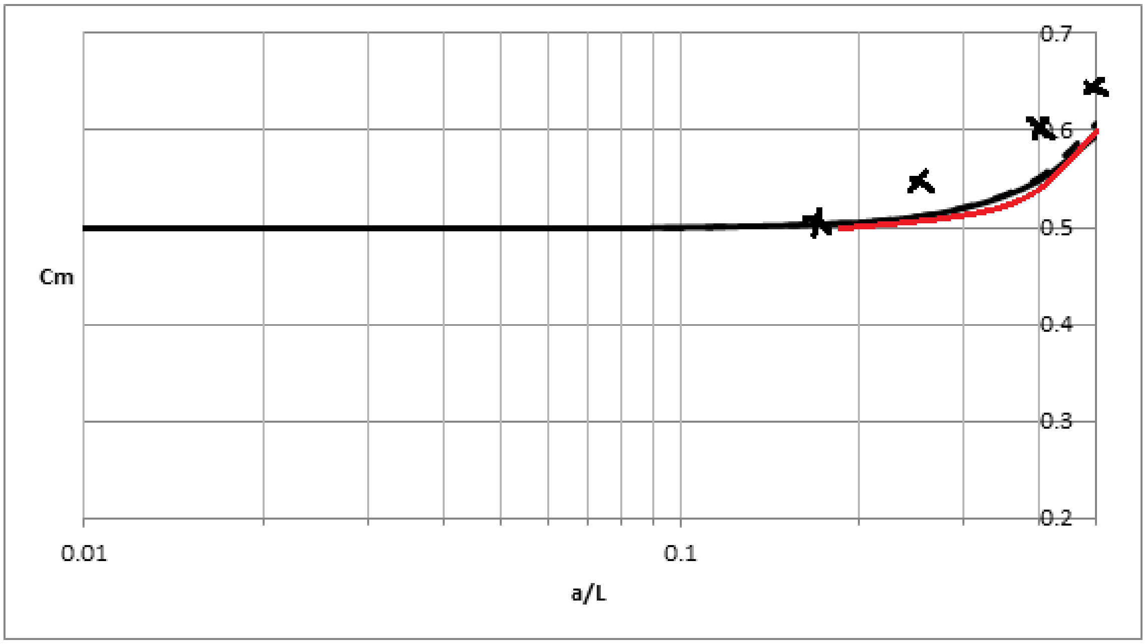

Fig. 5. Comparison between the present solution for the C_m of the two bubbles rising side by side (Eq. 18), solid line, van Wijngaarden [1] (Eq. 20) , dashed line, red line, Helfinstine and Dalton [9], crosses are the experimental data of Kendoush et al. [5].

For a bubble radius of $a = 0.57x10^{-3}$ m, $a/L = 0.2$ and $C_m = 0.4803$, Eq. (21) was solved by the integrating factor method. The properties of the working liquid are shown in Table 1. The solution was performed at different time steps, different separation, and different C_m calculated from the information of Figs. 4.16 and 4.17 of Sanada's thesis [15]. The analytical solution of Eq. (21) is the following

$$U_A = 0.1811[1 - \exp(-112.695t)] \quad \text{m/s} \tag{23}$$

Table 1 Physical properties of K1 fluid used by Sanada [15].

ρ (kg/m^3)	876
ν (m^2/s)	2.132x10^{-6}

Fig. 6 shows a comparison between the present solution of Eq. (23), and the experimental data of Sanada [15] where the agreement with the present solution is taken as an evidence of the internal consistency of the present theoretical calculations. The little hump in the experimental data of Sanada [15] is a form of experimental fluctuation.

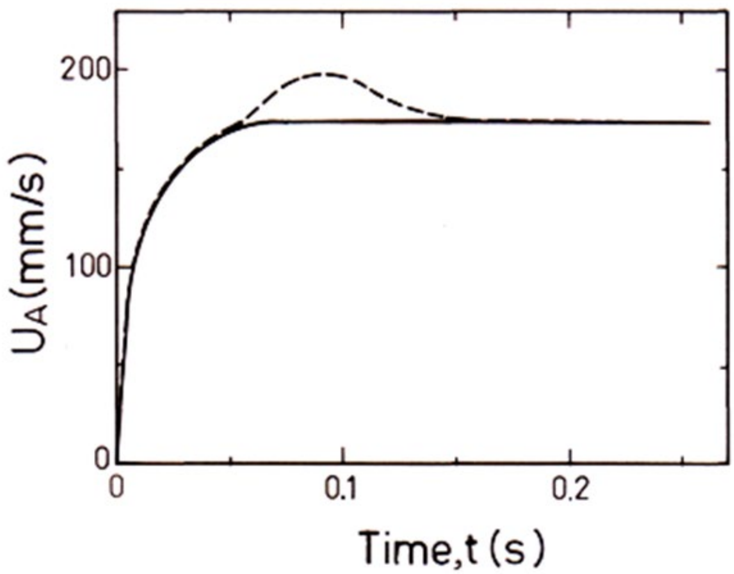

Fig. 6. Comparison of the present solution of Eq. (23) (——) with the experimental data of Sanada [15] (- - -) for the leading bubble.

4 Conclusions

New equations were derived in this paper for the virtual mass coefficients of two bubbles moving side by side and in tandem. The following characteristics were established from the new derivation of the virtual mass coefficients

- It increases with the decrease in the separation distance between two bubbles for the two bubbles rising side by side.
- It decreases with the decrease in the separation distance between two bubbles for the two bubbles rising in line.
- They approach asymptotically the value of 0.5 for the single spherical bubble when the separation distance between bubbles increases indefinitely. This is an evidence of the proof of the validity of the present analyses. The present solution was validated by comparison with theoretical solutions of other investigators, and the experimental data of the present author.

Acknowledgements

I wish to thank Prof. Sawsan A.M. Mohammed of the University of Baghdad for her helpful discussion. Thanks extended to Mr. Warren E. Overton of Augusta Technical College, Augusta, GA. for his help.

Funding

There was no funding associated with the completion of this work.

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Appendix : Derivation of the Velocity of Rise of Two Bubbles

A1. Two Bubbles rising in Line

The method of images was used by Kendoush [12] to derive the velocity potential of bubble A (Fig. 1) as given below

$$\phi_A = \frac{U_A}{2}\left[\frac{a^3}{r^2}\cos\theta + \frac{a^3}{L^2}\left(1+\frac{2r\cos\theta}{L}\right)+\left(\frac{a}{L}\right)^3\frac{a^3}{r^2}\cos\theta\right] \tag{A1}$$

The radial and tangential velocity components of the flow were obtained from $V = \nabla\phi$ as follows

$$V_r = U_A\cos\theta\left[-1+(a/r)^3-(a/L)^3+(a/L)^3(a/r)^3\right] \tag{A2}$$

and

$$V_\theta = U_A\sin\theta\left[1+0.5(a/r)^3+(a/L)^3+0.5(a/L)^3(a/r)^3\right] \tag{A3}$$

The drag force is calculated from the evaluation of the rate of viscous dissipation by using the following equation (Kendoush, [17])

$$W_A = \int_0^\pi (\tau_\theta V_\theta)_{r=a}\, 2\pi a^2 \sin\theta d\theta \tag{A4}$$

The tangential shear stress is given by the following equation

$$[\tau_\theta]_{r=a} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)+\frac{1}{r}\frac{\partial V_r}{\partial\theta}\right]_{r=a} \tag{A5}$$

Evaluating this equation and substituting it together with Eq. (A3) into Eq. (A4), we get the following after solving for the integral

$$W_A = 12\pi\mu a U_A^2[1+(a/L)^3]^2 \tag{A6}$$

The drag force on bubble A is given by $D_A = W_A/U_A$, which becomes equal to the following

$$D_A = 12\pi\mu a U_A[1+(a/L)^3]^2 \tag{A7}$$

A simple force balance on bubble A is done by equating the drag force from this equation to the buoyancy force as follows

$$12\pi\mu a U_A[1+(a/L)^3]^2 = (4/3)\pi a^3\rho g \tag{A8}$$

This equation yields Eq. (1) of the text.

Similarly for bubble B, the velocity potential is given by the following (Kendoush, 2007)

$$\emptyset_B = \frac{U_B}{2}\frac{a^3}{R^2}cos\beta + \frac{U_B}{2}\frac{a^6}{L^5}\left(1+\frac{2Rcos\beta}{L}\right)+\frac{U_B}{2}\left(\frac{a}{L}\right)^6\frac{a^3}{R^2}cos\beta \tag{A9}$$

and the drag force on bubble B becomes

$$D_B = 12\pi\mu a U_B[1+(a/L)^6]^2 \tag{A10}$$

A2. Two side-by-side bubbles

The velocity potential of bubble A (Fig. 2) was obtained by Kendoush [12] as follows

$$\emptyset_A = \frac{U_A}{2}cos\theta\left[\frac{a^3}{r^2}+\frac{r}{2}\left(\frac{a}{L}\right)^6+\frac{1}{4}\left(\frac{a}{L}\right)^6\frac{a^3}{r^2}\right] \tag{A11}$$

and for bubble B

$$\emptyset_B = \frac{U_B}{2}\frac{a^3}{R^2}cos\beta + \frac{U_B}{4}\left(\frac{a}{L}\right)^6 Rcos\beta + \frac{U_B}{8}\left(\frac{a}{L}\right)^6\frac{a^3}{R^2}cos\beta \tag{A12}$$

Here $U_A = U_B$. Following the same method of solution outlined in Section A1, we get Eq. (10) of the text.