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Well-balanced Unstructured Scheme for the Saint-Venant System with Topography

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Abstract- A well-balanced unstructured central-upwind scheme is proposed for the Saint-Venant System with topography. We prove that the designed scheme resolves the steady state solution (lake at rest). We illustrate the performance of the designed scheme using numerical tests. The obtained numerical results clearly demonstrate the well-balanced of the new scheme, its robustness and high resolution for shallow water equations with variable topography using the unstructured grids.

***Keywords*:** Shallow water, finite volume methods, well-balanced schemes.

1. Introduction

The central schemes are widely used after the pioneer work of Nessyahu and Tadmor (1990), where a second order finite volume central method on a staggered grid in space-time was proposed. This scheme offers higher resolution with simplicity of the Riemann-solver free approach. Following Kurganov and Tadmor (1999), this scheme suffers from excessive numerical viscosity when a small time step is considered. Kurganov et al. (2001) proposed the central-upwind schemes which use information about the propagation of the waves. Kurganov and Petrova (2005) extended the central-upwind scheme to triangular grids for solving two-dimensional systems of conservation laws. Bryson et al. (2011) proposed a central-upwind scheme for the Saint-Venant system on triangular grids with discontinuous bottom topography. In this paper we proposed a central-upwind scheme for the Saint-Venant System with discontinuous bottom topography on cell-vertex grids. The computational cells used in our method are the dual of the primary triangular mesh. We propose a well-balanced discretization of the source term due to the variable topography. The numerical examples demonstrate the ability of the proposed method to accurately resolve the small perturbations of the steady state solution.

The outline of the paper is as follows. In Section 2, we present the shallow water equations. In Section 3, we present the semi-discrete form of the proposed unstructured central-upwind scheme. The well-balanced discretization of the source term due to the variable topography is proposed in Section 4. Section 5 presents some numerical tests for our method. Some concluding remarks complete the study.

2. Shallow Water Equations

In this paper, we focus on the source term due to bottom topography in the following shallow water model:

$$
\begin{aligned}
& h_t + (hu)_x + (hv)_y = 0, \\
& (hu)_t + (hu^2 + \frac{1}{2} gh^2)_x + (huv)_y = -ghB_x, \\
& (hv)_t + (huv)_x + (hv^2 + \frac{1}{2} gh^2)_y = -ghB_y,
\end{aligned} \tag{1}
$$

where $B(x,y)$ is the topography, $h(x,y,t)$ is the water depth, $u(x,y,t)$ and $v(x,y,t)$ are the x - and y -components of the average velocity, and g is the gravity acceleration. The new vector of primitive variables $\mathrm{U} := (w, p, q)^T$ is used to rewrite the system (1) as follows:

$$
\mathrm{U}_t + \mathrm{F}_x + \mathrm{G}_y = \mathrm{S}, \tag{2}
$$

with

$$
\begin{aligned}
& \mathrm{F}(\mathrm{U}, B) = \left(p, \frac{p^2}{w-B} + \frac{1}{2} g (w-B)^2, \frac{pq}{w-B} \right)^T, \\
& \mathrm{G}(\mathrm{U}, B) = \left(q, \frac{pq}{w-B}, \frac{q^2}{w-B} + \frac{1}{2} g (w-B)^2 \right)^T, \\
& \mathrm{S}(\mathrm{U}, B) = \left(0, -g(w-B)B_x, -g(w-B)B_y \right)^T,
\end{aligned} \tag{3}
$$

where $w := h + B$ is the water surface elevation and $p := hu$ and $q := hv$ are the flow rates in $x-$ and $y-$directions.

3. The Unstructured Central-upwind Scheme

3.1. Unstructured Grid

First we consider an initial triangular discretization of the global domain $\mathcal{D}$ to obtain the computational cells denoted by M_j which are centered around the vertex as shown in Figure 1. We use the computational discretization $\mathcal{D} = \bigcup_{j=1}^{N} M_j$, where N is the number of cells which also represents the number of nodes of the initial triangular gird. To define the dual grid for each computational cell M_j around the node P_j, the center of mass of the surrounding triangles having P_j as a common vertex are connected to obtain the boundary ∂M_j of the computational cell M_j. The primitive variables are located at the center of mass G_j of each cell M_j of area $|M_j|$. The interfaces of this cell are denoted by $(\partial M_j)_1, (\partial M_j)_2, ...(\partial M_j)_{m_j}$ which are shared with the neighboring cells $M_{j1}, M_{j2}, ...M_{jm_j}$, respectively. The length of each cell-interface $(\partial M_j)_k$ is denoted by l_{jk} and its unit normal vector is denoted by $\vec{n}_{jk} := (\cos\theta_{jk}, \sin\theta_{jk})^T$, where θ_{jk} is the angle of the unit normal vector $\vec{n}_{jk}$ with the $x-$axis. The coordinates of G_j are denoted by $(x_j, y_j)^T$. The midpoint of the *kth* side of computational cell

M_j is denoted by P_{jk} and the two nodes of this side are denoted by P_{jk_s}, where $s=1,2$. The time step and the time at step n are denoted by Δt and $t^n := n\Delta t$, respectively.

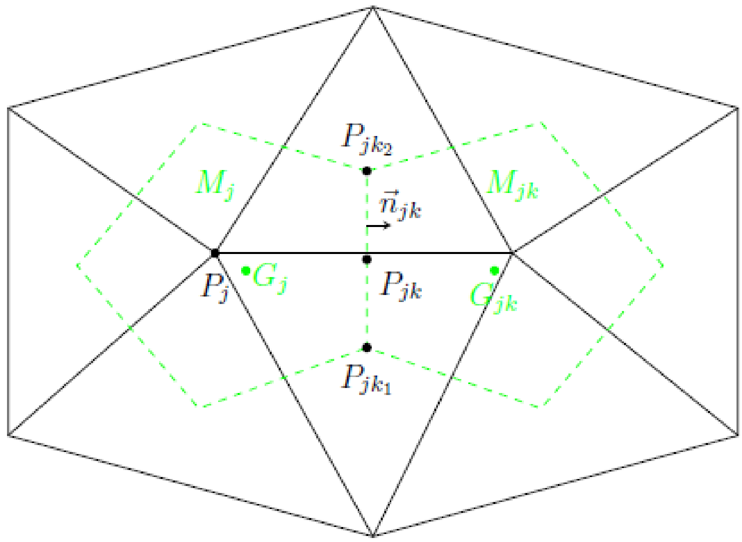

Fig.1. The unstructured grids used, solid lines are the initial grids and the dashed lines are the computational cells.

3.2. The Semi-discrete Form of the Unstructured Central-Upwind Scheme

The same method developed by Kurganov and Petrova (2005) for the central-upwind scheme on triangular grids can be used to obtain the following semi-discrete form of the central-upwind scheme on cell-vertex grids:

$$\begin{aligned}\frac{d\bar{U}_j}{dt} &= -\frac{1}{|M_j|}\sum_{k=1}^{m_j}\frac{l_{jk}\cos(\theta_{jk})}{a_{jk}^{in}+a_{jk}^{out}}[a_{jk}^{in}F(U_{jk}(P_{jk}),B_{jk})+a_{jk}^{out}F(U_j(P_{jk}),B_{jk})]\\ &-\frac{1}{|M_j|}\sum_{k=1}^{m_j}\frac{l_{jk}\sin(\theta_{jk})}{a_{jk}^{in}+a_{jk}^{out}}[a_{jk}^{in}G(U_{jk}(P_{jk}),B_{jk})+a_{jk}^{out}G(U_j(P_{jk}),B_{jk})]\\ &+\frac{1}{|M_j|}\sum_{k=1}^{m_j}l_{jk}\frac{a_{jk}^{in}a_{jk}^{out}}{a_{jk}^{in}+a_{jk}^{out}}[U_{jk}(P_{jk})-U_j(P_{jk})]+\bar{S}_j,\end{aligned} \tag{4}$$

where $\bar{U}_j$ is the approximation of the cell averages of the solution. The quantity $\bar{S}_j$ is a discretization of the cell averages of the source term which will be discussed in Section 4.

The semi-discrete form of the scheme (4) uses the bottom elevation B_{jk} at the midpoint of the *kth* cell interface, and the values $U_j(P_{jk})$ and $U_{jk}(P_{jk})$ at time t on the two sides of this interface are determined by using the reconstruction as explained in the next section.

The extreme values of the right- and left-sided local speeds at the *kth* interface of the computational cell M_j can be approximated as:

$$a_{jk}^{in} = -\min\{\lambda_1[V_{jk}(\mathrm{U}_j(P_{jk}))], \lambda_1[V_{jk}(\mathrm{U}_{jk}(P_{jk}))], 0\},$$
$$a_{jk}^{out} = \max\{\lambda_3[V_{jk}(\mathrm{U}_j(P_{jk}))], \lambda_3[V_{jk}(\mathrm{U}_{jk}(P_{jk}))], 0\}, \qquad (5)$$

where λ_1 and λ_3 are respectively the smallest and largest eigenvalues of the jacobian matrix V_{jk} of system (1).

3.3. Calculation of the Gradient

We use the Green-Gauss theorem to compute the gradient of the *ith* component of U_j, denoted by $\nabla \mathrm{U}_j^{(i)}$, $i = 1, 2, 3$:

$$\nabla \mathrm{U}_j^{(i)} = \frac{1}{|M_j|}\int_{M_j} \nabla \mathrm{U}_j^{(i)}(x,y)dxdy = \frac{1}{|M_j|}\sum_{k=1}^{m_j}\int_{(\partial M_j)_k} \tilde{\mathrm{U}}_{jk}^{(i)}\vec{n}_{jk}ds, \qquad (6)$$

where $\tilde{\mathrm{U}}_{jk}^{(i)}$ is the estimated value of $\mathrm{U}_j^{(i)}$ on the cell interface $(\partial M_j)_k$. In the proposed method we use the average values of the computational cell M_j and its neighbor M_{jk}. A modified minmod-type reconstruction is proposed. We compute m_j gradients and we select the one that has the smallest magnitude. This method is sufficient to avoid the numerical oscillations. For each $k \in [1, m_j]$, the gradient is calculated using equation (6) where the average values on each cell interface s are estimated by $\tilde{\mathrm{U}}_{js}^{(i)} = (\bar{\mathrm{U}}_j^{(i)} + \bar{\mathrm{U}}_{js}^{(i)})/2$, except for $s = k$ where we use an average of the values obtained for the interfaces attached to the interface k.

For each variable w, p and q, the above procedure leads to m_j gradient values and the one with smallest magnitude will be considered as the gradient of that variable in the cell. The values $\mathrm{U}_j(P_{jk})$ and $\mathrm{U}_{jk}(P_{jk})$ on the two sides of interfaces are obtained using the following linear reconstruction

$$\mathrm{U}_j(x,y) := \bar{\mathrm{U}}_j + \mathrm{U}_{jx}(x - x_j) + \mathrm{U}_{jy}(y - y_j). \qquad (7)$$

The topography at the vertices are used to construct the function $B(x,y)$. The elevation at the midpoint of the interface is defined as $B_{jk} = (B_{jk_1} + B_{jk_2})/2$. We consider the following reconstructed value of the bottom elevation at the center of the mass G_j:

$$B_j = \frac{1}{|M_j|}\int_{M_j} B(x,y)dxdy = \sum_{k=1}^{m_j}\mu_k B_{jk}, \qquad (8)$$

where $\mu_k = \mathcal{A}_{jk}/|M_j|$ and $\mathcal{A}_{jk}$ is the area of the triangle $G_j P_{jk_1} P_{jk_2}$ (see Figure 1).

The function $B(x,y)$ is defined as the union of m_j planes for each cell M_j. The *kth* plane passes through the bottom elevation B_j of G_j and the bottom elevations B_{jk_1} and B_{jk_2} at the two ends of the kth interface of M_j.

4. Well-balanced Discretization of the Source Term

The requirement for a well-balanced scheme is the preservation of the solution represented by $u \equiv 0$, $v \equiv 0$ and $w = C$, where C is a constant. For this type of solution, $\mathrm{U}_j(P_{jk}) = \mathrm{U}_{jk}(P_{jk}) = (C, 0, 0)^T$. This condition is sufficient to simplify the second and third equations of the semi-discrete form of the scheme (4) as follows:

$$
\begin{aligned}
&-\frac{g}{2\left|M_j\right|}\sum_{k=1}^{m_j} l_{jk}\cos(\theta_{jk})(C - B_{jk})^2 + \overline{\mathrm{S}}_j^{(2)} = 0,\\
&-\frac{g}{2\left|M_j\right|}\sum_{k=1}^{m_j} l_{jk}\sin(\theta_{jk})(C - B_{jk})^2 + \overline{\mathrm{S}}_j^{(3)} = 0.
\end{aligned}
\tag{9}
$$

The source term $\overline{\mathrm{S}}_j^{(2)}$ given by:

$$
\overline{\mathrm{S}}_j^{(2)} = -\frac{g}{\left|M_j\right|}\int_{M_j}(w - B)B_x dxdy,
$$

The following results are obtained by using the divergence theorem:

$$
\begin{aligned}
&\int_{M_j}(w - B)B_x dxdy = -\int_{M_j}\frac{1}{2}((w - B)^2)_x dxdy + \int_{M_j}(w - B)w_x dxdy\\
&= -\sum_{k=1}^{m_j}\int_{(\partial M_j)_k}\frac{1}{2}(w - B)^2\cos(\theta_{jk})ds + \int_{M_j}(w - B)w_x dxdy.
\end{aligned}
\tag{10}
$$

Therefore

$$
-\frac{g}{2\left|M_j\right|}\sum_{k=1}^{m_j} l_{jk}(w(P_{jk}) - B_{jk})^2\cos(\theta_{jk}) + \overline{\mathrm{S}}_j^{(2)} = -\frac{g}{\left|M_j\right|}\int_{M_j}(w - B)w_x dxdy.
\tag{11}
$$

For the source term $\overline{\mathrm{S}}_j^{(3)}$ we obtain the following result:

$$
-\frac{g}{2\left|M_j\right|}\sum_{k=1}^{m_j} l_{jk}(w(P_{jk}) - B_{jk})^2\sin(\theta_{jk}) + \overline{\mathrm{S}}_j^{(3)} = -\frac{g}{\left|M_j\right|}\int_{M_j}(w - B)w_y dxdy.
\tag{12}
$$

The discretization of the source term is well-balanced if the right-hand side terms of equations (11) and (12) are zero for $\mathrm{U} = (C, 0, 0)^T$. In the proposed method we use the following discretization of the source term

$$\bar{S}_j^{(2)} = \frac{g}{2|M_j|}\sum_{k=1}^{m_j} l_{jk}(w_j(P_{jk}) - B_{jk})^2 \cos(\theta_{jk}) - g(w_x)_j(\bar{w}_j - B_j),$$
$$\bar{S}_j^{(3)} = \frac{g}{2|M_j|}\sum_{k=1}^{m_j} l_{jk}(w_j(P_{jk}) - B_{jk})^2 \sin(\theta_{jk}) - g(w_y)_j(\bar{w}_j - B_j), \tag{13}$$

5. Numerical Tests

The proposed central-upwind scheme is employed to compute small perturbations of the “lake at rest" steady states with variable topography. The tests have been performed using $g = 1$. The third order total variation diminishing (TVD) Runge Kutta method (Shu and Osher, 1988) is used as temporal scheme. In this test, we evaluate the ability of the proposed scheme to simulate small perturbations of a stationary-state solution and we examine the well-balanced property of the scheme. The performance of the scheme is examined using a modified version of the test introduced by Le Veque (1998) using very small perturbations. We consider a rectangular computational domain $[0,2]\times[-0.5,0.5]$ and a variable topography defined as:

$$B(x,y) = 0.8\exp(-5(x-0.9)^2 - 50y^2). \tag{14}$$

Initially, the water surface is flat $w(x,y) = 1$ everywhere except in the rectangle domain defined by $0.05 < x < 0.15$

$$w(x,y) = \begin{cases} 1+\varepsilon, & 0.05 < x < 0.15, \\ 1. & \text{otherwise.} \end{cases} \tag{15}$$

Figure 2 shows the solution for the case $\varepsilon = 10^{-4}$ computed using an average cell area $|M_j| = 7.78\times10^{-6}$ at times $t = 0.6$ and $t = 1.8$. We do not observe any numerical oscillations and the proposed method presents a high resolution of small perturbations of the “lake at rest" steady states with variable topography. In order to show the importance of the proposed well-balanced discretization of the source term, we will use another approximation for this term by using the midpoint rule

$$\bar{S}_j^{(2)} = -g(\bar{w}_j - B_j)(B_x)_j,$$
$$\bar{S}_j^{(3)} = -g(\bar{w}_j - B_j)(B_y)_j, \tag{16}$$

where $(B_x)_j$ and $(B_y)_j$ are computed using the divergence theorem

$$(B_x)_j = \frac{1}{|M_j|}\sum_{k=1}^{m_j} l_{jk} B_{jk}\cos(\theta_{jk}),$$
$$(B_y)_j = \frac{1}{|M_j|}\sum_{k=1}^{m_j} l_{jk} B_{jk}\sin(\theta_{jk}). \tag{17}$$

Figure 3 (left) shows the 3D view of the solution at time $t = 0.6$ with $\varepsilon = 10^{-4}$ using the same mesh used for the proposed scheme. The non-well balanced scheme leads to spurious modes which influence the solution. Figure 3 (right) shows the solution, using the coarser mesh with average cell area $|M_j| = 2.24\times10^{-5}$, which is largely deformed.

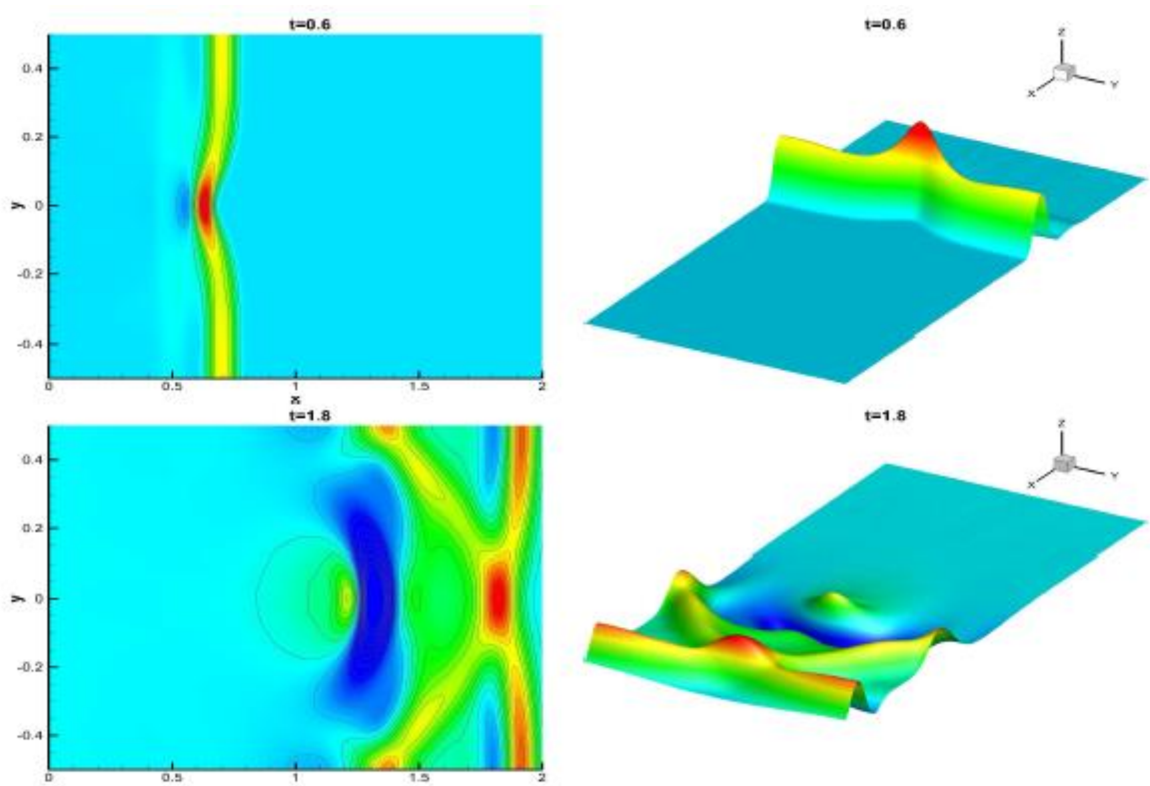

Fig. 2. Small perturbation computed by the well-balanced central-upwind scheme $\varepsilon = 10^{-4}$.

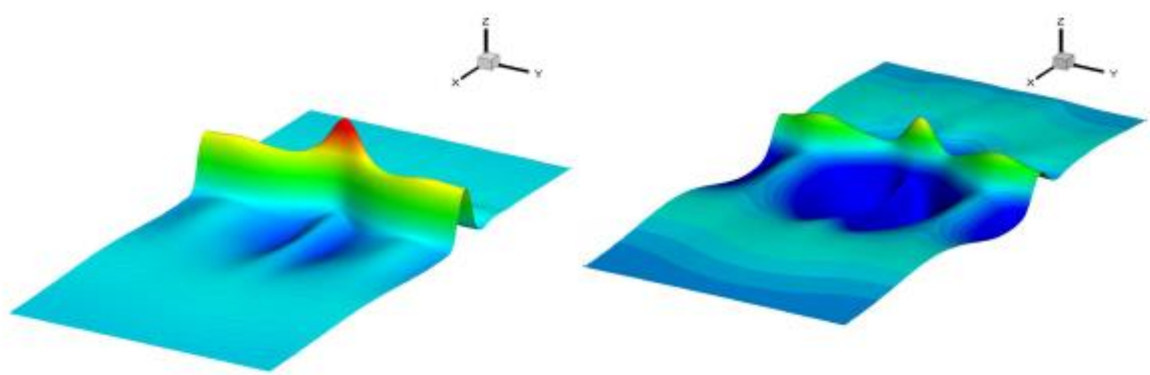

Fig. 3. Small perturbation $\varepsilon = 10^{-4}$: Left: non-well-balance using finer grid. Right: non-well-balance using coarse grid.

6. Conclusions

In this paper, we introduced a new well-balanced unstructured central-upwind scheme to approximate the solution of shallow water equations with variable bottom topography. The performance of the proposed method was tested using the small perturbations of the steady state solutions over variable topography. The numerical tests illustrate the ability of the proposed method to resolve the solution of the shallow water equations over variable topography. Currently we are working on the positivity of the scheme in order to develop a well-balanced positivity preserving unstructured central-upwind scheme for shallow water flows over variable topography.

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