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# A Nonlocal Elasticity Approach for the In-Plane Static Analysis of Nanoarches

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**Abstract** - Eringen's nonlocal elasticity theory is incorporated into classical beam model considering the effects of axial extension and the shear deformation to capture unique static behavior of the nanobeams under continuum mechanics theory. The governing differential equations are obtained for curved beams and solved exactly by using the initial value method. Circular uniform beam with concentrated loads are considered. The effects of shear deformation, axial extension, geometric parameters and small scale parameter on the displacements and stress resultants are investigated.

Keywords: Nanoarches, nonlocal elasticity, in-plane statics, exact solution, initial value method.

## 1. Introduction

Nano-sized beam structures have great potential applications in many different fields such as nanoscale actuation, sensing, and detection due to their remarkable mechanical, electronic and chemical properties. The growing interest in nanotechnology has fueled the study of nanostructures such as nanotrusses, nanobeams and nanoshells. Classical continuum mechanics cannot fully describe the mechanical behavior of these structures due to the absence of an internal material length scale in the constitutive law. Eringen's studies on nonlocal elasticity introduced integro-differential constitutive equations to account for the effect of long-range interatomic forces [1]. This theory states that the stress at a given reference point of a body is a function of the strain field at every point in the body; hence, the theory takes the long range forces between atoms and the scale effect into account in the formulation. Application of nonlocal elasticity for the formulation of nonlocal version of the Euler-Bernoulli beam model is initially proposed by Peddieson et al. [2]. Since then, the nonlocal theory, including nano-beam, plate and shell models were successfully developed using nonlocal continuum mechanics and many researchers reported on bending, vibration, buckling and wave propagation of nonlocal nanostructures [3-5]. Most of these studies focused on straight beam formulation, however, it is known that these structures might not be perfectly straight [6]. As an example, carbon nanotubes are long and bent, the bending being observed in isolated carbon nanotubes between electrodes or composite systems made from carbon nanotubes [7]. The curvature may be originated from buckling of axially loaded straight nanotubes or it is a result of fabrication and waviness affects the material stiffness. Although carbon nanotubes are usually not straight and have some waviness along its length, few investigations are known to be concerned with the vibration of these nanostructures.

In the study, in-plane static behavior of a planar curved nanobeam is investigated. Exact analytical solution of in-plane static problems of a circular nanobeam with uniform cross-section is presented. It is known that the size elimination of the nano scale effect may cause a significant deviation in the results. This study aims to overcome the problem by using Eringen's nonlocal theory. Initially, the governing differential equations of static behavior of a curved nanobeam are given by using the nonlocal constitutive equations of Eringen. The expressions for components of Laplacian of the symmetrical second order tensor in cylindrical coordinates given by Povstenko [8] are implemented in Eringen's nonlocal equations in order to obtain the governing equations of a curved beam in Frenet frame. Based on the initial value method, the exact solution of the differential equations is obtained. The displacements, rotation angle about the binormal axis and the stress resultants are obtained analytically. The axial extension and shear deformation effects are considered in the analysis. A parametric study is also performed to point out the effects of the geometric parameters such as slenderness ratio, opening angle, loading and boundary conditions. To the authors' best knowledge, almost all of the studies on the nonlocal beam theory has been discussed in the context of straight nanobeams. There is very limited number of papers on the curved nanobeams and most

of them neglect the effects of axial extension and shear deformation. They use numerical and approximate solution methods and consider only the nonlocal effect of bending moment. However, the results confirm a particular conclusion that bending deformation of the nano-cantilever beam subjected to a concentrated force reveals no nonlocal effect [9]. The present work will be helpful in the analysis and design of circular nanobeams with various combinations of loadings, boundary conditions and material properties.

## 2. Analysis

According to Eringen's nonlocal model, the stress values at a generic point are related to a weighted integral of strains over a certain domain. In isotropic media, it is assumed that a unique kernel weights all entries of stiffness tensor equally [1], and the equation is given as;

$$\sigma_{ij}^{nl}(\boldsymbol{x}) = \int_{\Omega} \alpha(\boldsymbol{x}, \boldsymbol{x}') \, \sigma_{ij}^{l}(\boldsymbol{x}') \, d\Omega \tag{1}$$

where  $\sigma_{ij}^{nl}$  and  $\sigma_{ij}^{l}$  are nonlocal and local stress tensors, respectively, x and x' are position vectors for two material points in domain  $\Omega$  and  $\alpha$  is a scalar kernel function. The integral constitutive equations of nonlocal elasticity can be simplified to an equivalent partial differential equation by making certain assumptions [1]:

$$(1 - \gamma^2 \nabla^2) \boldsymbol{\sigma}^{nl} = \boldsymbol{\sigma}^l \tag{2}$$

where  $\nabla^2$  is the Laplacian operator,  $\sigma^{nl}$  and  $\sigma^l$  are the nonlocal and local stress tensors, respectively, and  $\gamma = e_0 a$  is the nonlocal parameter that describes the effect of small scale on the mechanical behavior. The parameter  $e_0$  is a constant which has to be determined for each material independently and a is an internal characteristics length. Eringen estimated the parameter  $e_0$  as 0.39 [1]. Several authors reported that the value of  $e_0 a$  varies between 0 to 2 nm for analyzing carbon nanotubes [10-11]. The expressions for components of Laplacian of the symmetrical second order tensor are given by Povstenko [8]. Using these equations, the Laplacian of the nonlocal stress tensor  $\sigma^{nl}$  in cylindrical coordinates are obtained as follows:

$$\left(\nabla^2 \boldsymbol{\sigma}^{nl}\right)_{rr} = \nabla^2 \sigma_{rr}^{nl} - \frac{4}{r^2} \frac{\partial \sigma_{r\theta}^{nl}}{\partial \theta} - \frac{2}{r^2} \left(\sigma_{rr}^{nl} - \sigma_{\theta\theta}^{nl}\right) \tag{3}$$

$$\left(\nabla^2 \boldsymbol{\sigma}^{nl}\right)_{\theta\theta} = \nabla^2 \sigma_{\theta\theta}^{nl} + \frac{4}{r^2} \frac{\partial \sigma_{r\theta}^{nl}}{\partial \theta} + \frac{2}{r^2} \left(\sigma_{rr}^{nl} - \sigma_{\theta\theta}^{nl}\right) \tag{4}$$

$$\left(\nabla^2 \boldsymbol{\sigma}^{nl}\right)_{r\theta} = \nabla^2 \sigma_{r\theta}^{nl} - \frac{4}{r^2} \sigma_{r\theta}^{nl} + \frac{2}{r^2} \frac{\partial}{\partial \theta} \left(\sigma_{rr}^{nl} - \sigma_{\theta\theta}^{nl}\right)$$
(5)

$$\left(\nabla^2 \boldsymbol{\sigma}^{nl}\right)_{rz} = \nabla^2 \sigma_{rz}^{nl} - \frac{1}{r^2} \sigma_{rz}^{nl} - \frac{2}{r^2} \frac{\partial \sigma_{\theta z}^{hl}}{\partial \theta} \tag{6}$$

$$\left(\nabla^2 \boldsymbol{\sigma}^{nl}\right)_{\theta z} = \nabla^2 \sigma_{\theta z}^{nl} - \frac{1}{r^2} \sigma_{\theta z}^{nl} + \frac{2}{r^2} \frac{\partial \sigma_{rz}^{nl}}{\partial \theta}$$
(7)

$$\left(\nabla^2 \boldsymbol{\sigma}^{nl}\right)_{zz} = \nabla^2 \sigma_{zz}^{nl} \tag{8}$$

where,

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$
(9)

The Frenet coordinate system is used in the formulation of problems of curved beams. The cylindrical and Frenet coordinate systems are given in Figure 1. The stresses in a curved beam are given in the Frenet coordinate system as  $\sigma_t$ ,  $\sigma_{nt}$  and  $\sigma_{tb}$ . The remaining stresses ( $\sigma_n$ ,  $\sigma_b$  and  $\sigma_{nb}$ ) are assumed as zero. For in-plane problems of planar curved beams, the

stresses  $\sigma_t$ , and  $\sigma_{nt}$  and stress resultants  $F_n^{nl}$ ,  $F_t^{nl}$  and  $M_b^{nl}$  are considered. The relations between the stresses in Frenet and cylindrical coordinates are as follows:

$$\sigma_n = -\sigma_{rr} = 0 \; ; \quad \sigma_b = \sigma_{zz} = 0 \; ; \quad \sigma_t = \sigma_{\theta\theta} \neq 0 \tag{10}$$

$$\sigma_{nt} = -\sigma_{r\theta} \neq 0; \quad \sigma_{tb} = \sigma_{\theta z} \neq 0; \quad \sigma_{nb} = -\sigma_{rz} = 0$$
(11)

Since a uniform circular beam is considered in this study, the radius of the beam is constant, i.e.  $R(\theta) = R$ , and the coordinate *r* is described as (Figure 1);

$$r = R + \bar{r} \qquad \qquad \partial r = \partial \bar{r} \tag{12}$$

It is assumed that  $\bar{r}/R \ll 1$  (beam assumption).



Fig. 1: The Frenet and cylindrical coordinates of a curved beam.



Fig. 2: Circular beam with nonsymmetrical boundary and loading conditions.

The governing differential equations of nonlocal beams can be rewritten in the following form;

$$\frac{dw(\theta)}{d\theta} = u(\theta) + \frac{R}{EA} \left(1 + \frac{\gamma^2}{R^2}\right) F_t^{nl}(\theta)$$
(13)

$$\frac{du(\theta)}{d\theta} = -w(\theta) + R\Omega_b(\theta) + \frac{k_n R}{GA} \left(1 + \frac{\gamma^2}{R^2}\right) F_n^{nl}(\theta)$$
(14)

$$\frac{d\Omega_b(\theta)}{d\theta} = \frac{R}{EI_b} M_b^{nl}(\theta) \tag{15}$$

$$\frac{dM_b^{nl}(\theta)}{d\theta} = -RF_n^{nl}(\theta) \tag{16}$$

$$\frac{dF_t^{nl}(\theta)}{d\theta} = F_n^{nl}(\theta) \tag{17}$$

$$\frac{dF_n^{nl}(\theta)}{d\theta} = -F_t^{nl}(\theta) \tag{18}$$

where u and w are the normal and tangential displacements,  $\Omega_b$  is the rotation angle about the binormal axis,  $\theta$  is the angular coordinate; R is the radius of curvature of the beam; A is the cross-sectional area;  $I_b$  is the area moment of inertia of the cross-section with respect to the binormal axis;  $k_n$  is the factor of shear distribution along the normal axis;  $F_n^l$  and  $F_t^l$  are normal and tangential components of internal force, respectively;  $M_b^l$  is the internal moment about the binormal axis; E and G are respectively Young's and shear moduli. In this study, a circular beam with uniform doubly symmetric cross-section is considered.

Equations can also be stated in the matrix form as:

$$\frac{d\mathbf{y}(\theta)}{d\theta} = \mathbf{A}(\theta)\mathbf{y}(\theta) \tag{19}$$

where  $\mathbf{y}(\theta)$  is the vector of variables, namely,  $w, u, \Omega_b, M_b^{nl}, F_t^{nl}, F_n^{nl}, \mathbf{A}(\theta)$  is the 6×6 coefficient matrix. The solution of the Eq. (25) can be expressed as:

$$\mathbf{y}(\theta) = \mathbf{Y}(\theta, \theta_0) \mathbf{y}_0 \tag{20}$$

where  $\mathbf{Y}(\theta, \theta_0)$  is the fundamental matrix;  $\mathbf{y}_0 = \mathbf{y}(\theta_0)$  is the vector of initial values at the coordinate  $\theta_0$  (in this study  $\theta_0 = 0$ ).

The solution of this equation can be written in the following form:

$$\begin{bmatrix} w(\theta)\\ u(\theta)\\ \Omega_b(\theta)\\ M_b^{nl}(\theta)\\ F_t^{nl}(\theta)\\ F_n^{nl}(\theta) \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} & Y_{15} & Y_{16}\\ Y_{21} & Y_{22} & Y_{23} & Y_{24} & Y_{25} & Y_{26}\\ Y_{31} & Y_{32} & Y_{33} & Y_{34} & Y_{35} & Y_{36}\\ Y_{41} & Y_{42} & Y_{43} & Y_{44} & Y_{45} & Y_{46}\\ Y_{51} & Y_{52} & Y_{53} & Y_{54} & Y_{55} & Y_{56}\\ Y_{61} & Y_{62} & Y_{63} & Y_{64} & Y_{65} & Y_{66} \end{bmatrix} \begin{bmatrix} w_0\\ u_0\\ \Omega_{b0}\\ M_{b0}^{nl}\\ F_{t0}^{nl}\\ F_{t0}^{nl} \end{bmatrix}$$
(21)

where  $\mathbf{Y}(\theta, \theta_0)$  is the fundamental matrix which is obtained from the solution of the system of homogeneous equations, and  $\mathbf{y}_0 = \mathbf{y}(\theta_0)$  is the vector of initial values at the coordinate  $\theta_0$ , (in this study  $\theta_0 = 0$ ). The fundamental matrix satisfies the following requirements:

$$\mathbf{Y}(\theta_1, \theta_2)\mathbf{Y}(\theta_2, \theta_3) = \mathbf{Y}(\theta_1, \theta_3), \quad \mathbf{Y}(\theta_1, \theta_2) = \mathbf{Y}^{-1}(\theta_2, \theta_1)$$
$$\frac{d\mathbf{Y}(\theta, \theta_0)}{d\theta} = \mathbf{A}(\theta)\mathbf{Y}(\theta, \theta_0), \qquad \mathbf{Y}(\theta_0, \theta_0) = \mathbf{I}$$
(22)

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where I is unit matrix.

If the initial values,  $v_0$ ,  $\Omega_{n0}$ ,  $\Omega_{t0}$ ,  $M_{n0}^{nl}$ ,  $M_{t0}^{nl}$ ,  $F_{b0}^{nl}$  are known, the solution given in Eqn. (26) can be obtained analytically. These values can be solved from a system of linear equations which are obtained from the boundary conditions of the beam.

In this study, as a general case, a circular uniform beam with point loads at the coordinate ( $\theta = \theta_K$ ) is also investigated. This type of beam has two regions and the solutions for both regions are knows as:

$$\mathbf{y}_1(\theta_1) = \mathbf{Y}(\theta_1, \theta_0) \mathbf{y}_{10} \quad \text{for} \quad -\theta_A \le \theta_1 \le \theta_K$$
(23)

$$\mathbf{y}_2(\theta_2) = \mathbf{Y}(\theta_2, \theta_K) \mathbf{y}_{2K} \quad \text{for} \quad \theta_K \le \theta_1 \le \theta_B \tag{24}$$

where  $\mathbf{y}_{2K}$  is the vector of initial values for the second region at coordinate  $\theta_K$ . In order to calculate twelve components of the vectors  $\mathbf{y}_{10}$  and  $\mathbf{y}_{2K}$ , twelve equations of boundary and continuity conditions are used. The continuity condition at that point is:

$$\mathbf{y}_1(\boldsymbol{\theta}_K) + \mathbf{K} = \mathbf{y}_{2\mathbf{K}} \tag{25}$$

where  $\mathbf{K} = \begin{bmatrix} 0 & 0 & M_{Kn} & M_{Kt} & F_{Kb} \end{bmatrix}^{\mathrm{T}}$  is the loading vector. Thus, Eqn. (30) is rewritten as:

$$\mathbf{y}_{2}(\theta_{2}) = \mathbf{Y}(\theta_{2}, \theta_{K})\mathbf{y}_{1}(\theta_{K}) + \mathbf{Y}(\theta_{2}, \theta_{K})\mathbf{K}$$
(26)

By substituting Eqn. (29) into Eqn. (32), the following equation is obtained:

$$\mathbf{y}_{2}(\theta_{2}) = \mathbf{Y}(\theta_{2}, \theta_{K})\mathbf{Y}(\theta_{K}, \theta_{0})\mathbf{y}_{10} + \mathbf{Y}(\theta_{2}, \theta_{K})\mathbf{K}$$
(27)

Eqn. (33) can be rewritten by using Eqn. (28) as follows:

$$\mathbf{y}_{2}(\theta_{2}) = \mathbf{Y}(\theta_{2}, \theta_{0})\mathbf{y}_{10} + \mathbf{Y}(\theta_{2}, \theta_{0})\mathbf{Y}^{-1}(\theta_{K}, \theta_{0})\mathbf{K}$$
(28)

Then, the analytical functions of the displacements, rotation angle of the cross-section and the force resultants for both region can be obtained.

## 3. Clamped-clamped circular nanobeam loaded at midspan

The effects of several parameters on the static behavior of a circular nanobeam with clamped ends are studied in this example. The beam is loaded by a normal force  $F_0$  at its midspan (Figure 3).

The ratio of nonlocal and local displacements  $u_0/u_{0L}$  and moments  $M_{b0}/M_{b0L}$  at the midspan are obtained for several parameters. The effects of small scale parameter  $R/\gamma$ , slenderness ratio  $\lambda$  and opening angle  $\theta_t$  on the displacement ratio  $u_0/u_{0L}$  and moment ratio  $M_{b0}/M_{b0L}$  at the midspan are studied.



Fig. 3: Clamped-clamped circular nanobeam loaded at the midspan ( $\theta_t = 120^o$ ).

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Figure 4a shows the displacement ratio against the small scale parameter  $R/\gamma$  for the beam with the opening angle of  $\theta_t = 120^\circ$  and different slenderness ratios  $\lambda = 50,100$  and 150. The effect of small scale parameter  $R/\gamma$  on the displacement ratio  $u_0/u_{0L}$  is more significant for smaller slenderness values. This effect attenuates if the opening angle of the beam is decreased (i.e. the curves representing the displacement ratio becomes closer for different slenderness ratio). It is observed that, the small scale effect becomes more important for a slender beam with considerably small opening angle. The effects of axial extension and shear deformation on the displacement are studied for several values of opening angle and slenderness ratio. For the brevity, only the results for a beam with opening angle of  $\theta_t = 120^\circ$  and slenderness ratio of  $\lambda = 50$  is given in Figure 4b.



Fig. 4: The effect of  $R/\gamma$  on the ratio of local and nonlocal displacements  $u_0/u_{0L}$  for a clamped-clamped beam with  $\theta_t = 120^o$ (a) For different values of  $\lambda$  (b) For different effects.

The difference between the results of the cases (i.e. considering axial extension or considering shear deformation) increases with the increasing slenderness. From the figure, one can see that the axial extension has the dominant effect for all opening angles and slenderness ratio. The beam theory neglecting the effects of axial extension and shear deformation gives acceptable results for only a slender and deep curved beam where the bending deformation is the main effect. Moreover, when the beam is stubby, the shear deformation effect becomes also significant. The displacements for the case neglecting all effects (i.e. only the nonlocal effects of bending moment is considered) are same for both local and nonlocal theories. Similar result is obtained by Li [9] for straight beams with concentrated loads.

Figure 5a gives the diagram of the moment ratio  $M_{b0}/M_{b0L}$  against the small scale parameter  $R/\gamma$  for the beam with the opening angle of  $\theta_t = 120^{\circ}$  and slenderness ratio of  $\lambda = 50, 100$  and 150. The difference between the results of the cases (i.e. considering axial extension or considering shear deformation) increases with the increasing slenderness. This result shows that the axial extension is the main effect on the displacement ratio. Moment ratio increases with the increasing slenderness ratio for larger opening angle and the curves obtained for different slenderness ratio become closer with the decreasing opening angle.



Fig. 5: The effect of  $R/\gamma$  on the ratio of local and nonlocal moments  $M_{b0}/M_{b0L}$  for a clamped-clamped beam with  $\theta_t = 120^o$ (a) For different  $\lambda$  values (b) For different effects.

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The results of the cases considering or neglecting the axial extension and shear deformation effects for a beam with opening angle of  $\theta_t = 120^o$  and slenderness ratio of  $\lambda = 50$  is given in Figure 5b. Axial extension is the main contributing effect for all opening angle, as expected. Moment ratio increases with decreasing opening angle for all slenderness ratio.



Fig. 6: Displacements obtained by local and nonlocal theories  $(R/\gamma = 1)$ (a)  $\theta_t = 120^o$ ,  $\lambda = 100$ ; (b)  $\theta_t = 30^o$ ,  $\lambda = 100$ ; (c)  $\theta_t = 30^o$ ,  $\lambda = 100$ .

As it is well known from the local theory, the Euler beam theory gives acceptable results for a slender and deep curved beam, but the results are not reasonable when the curved beam is shallow. Deep and shallow curved beams exhibit different static and dynamic behavior [12-13]. A shallow curved beam deforms along a different path representing another characteristic deformed shape.

# 4. Conclusion

A new size-dependent general beam theory is presented within the framework of Eringen's nonlocal elasticity theory for static behavior of curved nanobeams. Nonlocal constitutive equations are implemented in the classical beam equations. Axial extension and shear deformation effects and their size-dependent effects along with the size-dependent effects of bending moment are incorporated in the analytical model. Initial value method is used for the exact solution and the results are obtained analytically. While only uniform circular nanobeams bearing concentrated loads are investigated in this paper, the equations can easily be expanded to provide sufficient generality in the choice of loading and geometry. Based on the analysis presented here, it is also possible to investigate the dynamics of curved nanobeams. Also, for engineering applications, it may be possible to develop an exact nonlocal beam finite element.

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